ON THE DISTRIBUTION OF THE LENGTH OF A SPHERICAL RANDOM VECTOR

BY

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To my grandmother

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This dissertation is concerned with determining the probability density function of the length of a spherical random vector.

The case of the sum of independent identically distributed random vectors, each of which has a uniform distribution on the unit m-sphere is first investigated. Results which are similar to those for the three dimensional situation are established. The probability density function of the length of the sum is given in terms of the probability density function of the absolute value of a single element of that vector.

The next case is to investigate the general family of spherical densities. We do so by two methods. First we use the characteristic function to develop the main result for an odd number of dimensions. Then we derive an integral equation which is useful for all values of m ($m \geq 2$), the number of dimensions. We use this integral equation to reproduce some known results. It is made clear that the results established for the uniform distribution on the unit sphere is a special case of this more general result.

CHAPTER I
INTRODUCTION

1.1 Directional Data

Quite often scientific data are recorded as unit vectors. As such, these vectors can be considered as directions in space and are referred to as directional data. In two dimensions these vectors correspond to points on the unit circle. In three dimensions these points are located on the surface of a sphere. For higher dimensions we speak of these points as lying on the surface of a hypersphere.

The study of directional data was first focused on the two dimensional and three dimensional situation. However, one can derive results for a general number of dimensions using the same techniques which are required for the study of two or three dimensions; this is easily seen in the case of a random vector which has a uniform distribution on the sphere.

Results pertaining to higher than three dimensions are of practical importance, since data may be collected on several variables. A vector corresponding to information on each of these variables may be scaled so that the length of the vector is unity.

The usual methods for analyzing multivariate data are based on the multivariate Gaussian distribution, but these methods are not applicable here and so an investigation of distributions on the sphere is of interest to statisticians.

Mardia (1972) is a good introduction to the literature (prior to 1972) on directional data. Watson (1982) deals in detail with a random vector which has a uniform distribution on the unit sphere. This book provides an indication of the direction in which research in this area is going and it references many of the recent contributions to the subject.

Directional data have been useful in diverse areas. Amongst these are sampling, astronomy, medicine, psychology, physics, biology, geography and meteorology. Some of the main models used to analyze directional data are the Bingham distribution, the uniform distribution on the unit sphere, the Langevin distribution, Saw's distribution and the von Mises distribution on the circle.

In this chapter we study the uniform distribution on the unit m-sphere. We consider an arbitrary number of dimensions. In Section 1.2 we introduce the notation which we use throughout this chapter. Section 1.3 contains some distributional results for a random vector which has a uniform distribution on the unit m-sphere. Section 1.4 is a summary of a paper by Saw (1986).

1.2 Notation

In what follows R^m denotes m-dimensional Euclidean space.

With $\|\underline{x}\| = (\underline{x}'\underline{x})^{\frac{1}{2}}$ let $S_m=\{\underline{x} \in R^m: \|\underline{x}\| = 1\}$ be the unit m-sphere and $B_m = \{\underline{x} \in R^m:\|\underline{x}\| \leq 1\}$ be the unit m-ball.

If W is a random variable with probability density f(w) given by

$$f(w) = \begin{cases} \dfrac{(1-w^2)^{\nu-\frac{1}{2}}}{B(\frac{1}{2},\nu+\frac{1}{2})} & -1<w<1, \\ 0, & \text{elsewhere}, \end{cases}$$

we will write $W \sim W^{\nu}$. Throughout, we let $\nu = \frac{1}{2}m-1$. Here $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ is the usual beta function. If $P(X<t)=P(Y<t)$ $\forall$ $t \in R^1$ we write $X \sim Y$. For a random variable having a probability density given by

$$g(x) = \begin{cases} \dfrac{x^{a-1}(1-x)^{b-1}}{B(a,b)} & 0<x<1 \\ 0 & \text{elsewhere}, \end{cases}$$

we write $X \sim B(a,b)$.

Wilks (1962) gives the gamma distribution as a one parameter family; we adopt the same convention here. Thus if Y has a gamma density, then the probability density function is given by

$$g(y) = \begin{cases} \dfrac{y^{\nu-1}e^{-y}}{\Gamma(\nu)} & 0<y<\infty, \\ 0 & \text{elsewhere}, \end{cases}$$

and we write $Y \sim G(\nu)$. As usual, χ_m^2 denotes a chi square random variable with m degrees of freedom and χ_m is the positive square root of a χ_m^2 random variable. If
$f(y_1, \ldots, y_k) =$

$$\frac{\Gamma(\nu_1+\nu_2+\ldots+\nu_{k+1})}{\Gamma(\nu_1)\ \Gamma(\nu_2)\ldots\Gamma(\nu_k)} y_1^{\nu_1-1}\ldots y_k^{\nu_k-1}(1-y_1-\ldots-y_k)^{\nu_{k+1}-1},$$

where $0<y_i<1$, $i=1,2,\ldots,k$ and $0<y_1+y_2+\ldots+y_k<1$. We say $y_1,y_2,\ldots,y_k$ have a Dirichlet distribution, denoted by $D(\nu_1,\nu_2,\ldots, \nu_k: \nu_{k+1})$.

If a random vector $\underline{Y}$ has a multivariate normal distribution with mean vector $\underline{\mu}$ and variance-covariance matrix Σ, we write this as $\underline{Y} \sim N(\underline{\mu}, \Sigma)$.

We define the function $\Lambda_\nu(t)$ by the infinite series

$$\Lambda_\nu(t) = \sum_{k=0}^{\infty} (\tfrac{1}{4}t^2)^k \frac{\Gamma(\nu+1)}{k!\Gamma(\nu+k+1)}$$

$$= \sum_{k=0}^{\infty} (\tfrac{1}{2}t^2)^k \frac{1}{k!\ m(m+2)\ldots(m+2k-2)}.$$

The function $\Lambda_\nu(t)$ is related to the Bessel function $J_\nu(t)$ and the modified Bessel function $I_\nu(t)$ by

$$I_\nu(t) = (\tfrac{1}{2}t)^\nu \frac{\Lambda_\nu(t)}{\Gamma(\nu+1)},$$

and

$$J_\nu(t) = (\tfrac{1}{2}t)^\nu \frac{\Lambda_\nu(it)}{\Gamma(\nu+1)}.$$

The coefficient of a^n, in the power series expansion of the function $(1-2at+a^2)^{-\nu}$, is $C_n^\nu(t)$ and is called Gegenbauer polynomial of degree n. The fact that $C_0^\nu(t)=1$ will be useful. Finally, the confluent hypergeometric function $M(a,b:z)$ is defined by the infinite series

$$M(a,b:z) = \sum_{k=0}^{\infty} \frac{z^k (a)_k}{k!\,(b)_k},$$

where $(a)_k$ in Pochammer's symbol and is $a(a+1)\dots(a+k-1)$ and $(a)_0 = 1$.

1·3 The Uniform Distribution on the Unit Sphere

For $-1<t<1$ and $\underline{\zeta} \in S_m$, let $C_m(\underline{\zeta},t)=\{\underline{u} \in S_m : \underline{\zeta}'\underline{u}>t\}$ and $\mathcal{A}$ be the smallest σ-algebra generated by $\{C_m(\underline{\zeta},t): \underline{\zeta}\in S_m, t \in [-1,1]\}$; thus,

$$\mathcal{A} = \sigma\,\{C_m(\underline{\zeta},t): \underline{\zeta} \in S_m,\ t \in [-1,1]\}.$$

Furthermore let U_m be the measure given by

$$U_m\{C_m(\underline{\zeta},t)\} = \frac{\text{Surface area of } C_m(\underline{\zeta},t)}{\text{Surface area of } S_m}.$$

Transformation by multiplying by an orthogonal matrix is a rotation of the original figure and thus areas and angles in the original figure are preserved under such a transformation. It follows that for $A \varepsilon \mathcal{A}$ and for P, an orthogonal matrix, we have $U_m(A) = U_m(PA)$, and consequently $U_m(C_m(\underline{\zeta},t))$ does not depend on $\underline{\zeta}$. The triplet $(S_m, \mathcal{A}, U_m)$ is a probability space. We can construct any number of densities on the unit sphere S_m with respect to the measure U_m. However in this dissertation we restrict our attention mainly to the density

$$f(\underline{u}) = 1 \qquad \underline{U} \in S_m ,$$

we say $\underline{U}$ has a uniform distribution on the unit m-sphere and write $\underline{U} \sim U(S_m)$.

We can now give a characterization of a random vector $\underline{U}$ having a uniform distribution on the unit sphere.

Theorem 1. Let $\underline{Z}$ be an mx1 component vector of independent identically distributed Gaussian random variables with $\mathcal{E}\, Z_1 = 0$ and $\mathcal{E}\, Z_1^2 = 1$. Then

(i) $\frac{\underline{Z}}{\|\underline{Z}\|} \sim U(S_m)$,

(ii) $\frac{\underline{Z}}{\|\underline{Z}\|}$ and $\|\underline{Z}\|$ are independent, and

(iii) $\|\underline{Z}\|^2 \sim \chi_m^2$.

Proof: Make the transformation $R=\|\underline{Z}\|$ and $\underline{U} = \frac{\underline{Z}}{\|\underline{Z}\|}$. The Jacobian of this transformation is R^{m-1}. Thus

$$f(r,\underline{u}) = \frac{r^{m-1}}{2^{\frac{1}{2}(m-1)}} \cdot \frac{e^{-\frac{1}{2}r^2}}{\Gamma(\frac{1}{2}m)} \cdot \frac{\Gamma(\frac{1}{2}m)}{2\left(\Gamma(\frac{1}{2})\right)^m},\quad 0<r<\infty,\ \|\underline{U}\| = 1,$$

which is the result we set out to prove.

This theorem is very useful. We can obtain results about the components of a random vector $\underline{U}$ having a uniform distribution of the unit sphere by examining the corresponding components of a vector $\frac{\underline{Z}}{\|\underline{Z}\|}$ where $\underline{Z}\sim N(\underline{0},I)$. Our familiarity with $\underline{Z}$ greatly simplifies the task.

Result 1. For every fixed mxm orthogonal matrix P, we have $P\underline{U}\sim U(S_m)$.

Proof: Theorem 1 tell us that $\underline{U}\sim\frac{\underline{Z}}{\|\underline{Z}\|}$ where $\underline{Z}\sim N(\underline{0},I)$. It follows that $P\underline{U}\sim\frac{P\underline{Z}}{\|\underline{Z}\|}$. Making the transformation $\underline{Y} = P\underline{Z}$ and noting that $\frac{P\underline{Z}}{\|\underline{Z}\|} = \frac{P\underline{Z}}{\sqrt{(\underline{Z}'P'P\underline{Z})}} = \frac{\underline{Y}}{\|\underline{Y}\|}$ we have $P\underline{U}\sim \frac{\underline{Y}}{\|\underline{Y}\|}$ where $\underline{Y}\sim N(\underline{0},I)$.

Result 2. For every fixed unit vector $\underline{\alpha}$, $\underline{\alpha}'\underline{U}\sim W^{\nu}$.

Proof: Without loss of generality we can consider $\underline{\alpha}'$ as the first row of the matrix P in Result 1. It is now obvious that $\underline{\alpha}'\underline{U} \sim \frac{Y_1}{\|\underline{Y}\|}$. We know that $\frac{Y_1^2}{\|\underline{Y}\|^2} \sim B(\frac{1}{2}, \frac{1}{2}(m-1))$. The result is immediate on making the square root transformation and on noting that $\frac{-Y_1}{\|\underline{Y}\|} \sim \frac{Y_1}{\|\underline{Y}\|}$.

We now give a more general version of Result 2; but we first give as lemmas, two results from Wilks.

Lemma 1. Suppose $X_1, X_2, \ldots, X_{k+1}$ independent random variables having gamma distributions $G(\nu_1), \ldots, G(\nu_{k+1})$. Make the transformation,

$$Y_i = \frac{X_i}{X_1 + X_2 + \ldots + X_{k+1}} \qquad i=1,2,\ldots,k,$$

then $Y_1, Y_2, \ldots, Y_k$ has the k-variate Dirichlet distribution $D(\nu_1, \nu_2, \ldots, \nu_k : \nu_{k+1})$.

Lemma 2. If $(X_1, X_2, \ldots, X_k)$ is a random vector having the k-variate Dirichlet distribution then the marginal distribution of $(X_1, X_2, \ldots, X_{k_1})$, $k_1 < k$ is the k-variate Dirichlet distribution $D\ (\nu_1, \ldots, \nu_{k_1} : \nu_{k_1+1} + \ldots + \nu_{k+1})$.

Result 3. Partition $\underline{U}'$ as $(\underline{U}'_1, \underline{U}'_2)$ where $\underline{U}'_1 = (U_1, U_2, \ldots, U_k)$ and $\underline{U}'_2 = (U_{k+1}, \ldots, U_m)$ then

(i) $\frac{\underline{U}_1}{\|\underline{U}_1\|} \sim U(S_k)$,

(ii) $\frac{\underline{U}_1}{\|\underline{U}_1\|}$ and $\|\underline{U}_1\|$ are independent,

(iii) $\|\underline{U}_1\|^2 \sim B(\frac{1}{2}k, \frac{1}{2}(m-k))$ and

(iv) The probability density of $\underline{U}_1$, say $g(\underline{u}_1)$, is given by

$$g(\underline{u}_1) = \frac{\Gamma(\frac{1}{2}m)}{(\Gamma(\frac{1}{2}))^k\Gamma(\frac{1}{2}(m-k))} (1 - \underline{u}_1'\underline{u}_1)^{\frac{1}{2}k-1} \qquad 0 < \underline{u}_1'\underline{u}_1 < 1,$$

Proof: Equation (i) is immediate on noting that

$$\frac{\underline{U}_1}{\|\underline{U}_1\|} \sim \frac{\underline{Z}_1}{\|\underline{Z}\|} \cdot \frac{\|\underline{Z}\|}{\|\underline{Z}_1\|} = \frac{\underline{Z}_1}{\|\underline{Z}_1\|},$$

and by Theorem 1 we obtain the result that $\frac{\underline{U}_1}{\|\underline{U}_1\|}$ has a uniform distribution on the unit k-sphere.

Now $\frac{\underline{U}_1}{\|\underline{U}_1\|} \sim \frac{\underline{Z}_1}{\|\underline{Z}_1\|}$ and by Theorem 1, we have that $\frac{\underline{Z}_1}{\|\underline{Z}_1\|}$ is independent of $\|\underline{Z}_1\|$. It follows that $\frac{\underline{Z}_1}{\|\underline{Z}_1\|}$ is independent of $\|\underline{Z}\|$, and consequently $\frac{\underline{Z}_1}{\|\underline{Z}_1\|}$ is independent of $\frac{\|\underline{Z}_1\|}{\|\underline{Z}\|}$ which is a restatement of (ii).

Equation(iii) is immediate on noting that

$$\|\underline{U}_1\|^2 \sim \frac{\sum_{i=1}^{k} Z_i^2}{\sum_1^k Z_i^2 + \sum_{k+1}^{m} Z_i^2},$$

where $\underline{Z} \sim N(\underline{0}, I)$. It is clear that Z_i^2 $i=1,2,...,m$ are independent χ_1^2 random variables. In the notation of Wilks $\frac{1}{2}Z_i^2 \sim G(\frac{1}{2})$, and on using Lemma 1 it is clear that $\frac{Z_i}{\|\underline{Z}\|}$,

$i = 1,...,m-1$ have a $D(\frac{1}{2},\frac{1}{2},...,\frac{1}{2}:\frac{1}{2})$ distribution. Lemma 2 tells us that the random variables $\frac{Z_i}{\|\underline{Z}\|}$ $i=1,...,k$ have a Dirichlet distribution $D(\frac{1}{2},\frac{1}{2},...,\frac{1}{2}: \frac{m-k}{2})$. On making the symmetric square root transformation and on multiplying by the Jacobian of this transformation we see that $\frac{\underline{Z}_1}{\|\underline{Z}_1\|}$ has the density given in (iv).

Result 4. Let $\underline{U}$ be a vector having a uniform distribution on the unit sphere, then

(i) $\mathcal{E}\,\underline{U} = \underline{0}$ and

(ii) $\mathcal{E}\,\underline{U}\,\underline{U}' = \frac{1}{m} I$.

Proof: For $j= 1,2,...,m$ it is clear that

$$0 = \mathcal{E}\, Z_j = \mathcal{E}\, \frac{Z_1}{\|\underline{Z}\|} \cdot \|\underline{Z}\| = \mathcal{E}\, \frac{Z_1}{\|\underline{Z}\|}\; \mathcal{E}\, \|\underline{Z}\| \;.$$

It follows that

$$\mathcal{E}\ U_j = \mathcal{E}\ \frac{Z_j}{\|Z\|} = 0.$$

Furthermore

$$\delta_{ij} = \mathcal{E}\ Z_i Z_j = \mathcal{E}\ \frac{Z_i}{R}\cdot\frac{Z_j}{R}\ .\ R^2 = \mathcal{E}\ \frac{Z_i}{R}\frac{Z_j}{R}\cdot\mathcal{E}\ R^2\ ,$$

where $$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j. \end{cases}$$

Hence, $\mathcal{E}\ U_i U_j = \dfrac{\delta_{ij}}{\mathcal{E}\ R^2}$ and $\mathcal{E}\ R^2 = m$, and the proof is complete.

From the multivariate central limit theorem, Rao (1965) section 2c.5, it is clear that if $\underline{U}_i$ i=1,...,n are independent and identically distributed random vectors with $\underline{U}_1 \sim U(S_m)$, then

$$\sqrt{(m/n)}\ \sum_{i=1}^{n}\underline{U}_i \xrightarrow{d} N(\underline{0},\ I)$$

and

$$(m/n)\ \left(\sum_{i=1}^{n}\underline{U}_i\right)'\left(\sum_{i=1}^{n}\underline{U}_i\right) \xrightarrow{d} \chi^2_m$$

Result 5. The characteristic function $\phi(t)$, and the moment generating function M(t) are given by

$$\phi(\underline{t}) = \mathcal{E}\ e^{i\underline{t}'\underline{U}} = \Gamma(\nu+1)\left(\tfrac{1}{2}\|\underline{t}\|\right)^{-\nu} J_\nu(\|\underline{t}\|)$$

and $M(\underline{t}\ ) = \mathcal{E}\ e^{\underline{t}'\underline{U}} = \Gamma(\nu+1)\left(\frac{1}{2}\|\underline{t}\|\right)^{-\nu} I_\nu(\|\underline{t}\|)$.

Proof: Any real vector $\underline{t}$ can be written as $\|\underline{t}\|\ \underline{\alpha}$ where $\underline{\alpha}$ is a unit vector in the direction of $\underline{t}$. The result

$$\mathcal{E}\ e^{i\underline{t}'\underline{U}} = \mathcal{E}\ e^{i\|\underline{t}\|\underline{\alpha}'\underline{U}} = \int_{-1}^{1} e^{i\|\underline{t}\|w}\ \frac{(1-w^2)^{\nu-\frac{1}{2}}}{B(\frac{1}{2},\ \nu+\frac{1}{2})}dw$$

$$=\Gamma(\nu+1)\ \left(\tfrac{1}{2}\|\underline{t}\|\right)^{-\nu} J_\nu(\|\underline{t}\|)$$

is now immediate from Result 3 and formula (8.4.10) of Gradshteyn and Ryzhik(1980). The corresponding result for the moment generating function is now obvious by Result 2 and formula (8.43.1).

This result is somewhat more significant than it appears. If we partition $\underline{U}$ into $\underline{U}_1$ and $\underline{U}_2$ as we did in Result 3, it should be clear that we can determine the characteristic function and moment generating function of $\underline{U}_1$ from the above result. If we let $\underline{a}_1$ be a kx1 vector, we can write $\underline{a}'$ as $\underline{a}'=(\underline{a}',\underline{0}')$ where $\underline{0}'$ is an 1xm-k vector of zeros, thus $\Lambda_\nu(\|\underline{a}\|) = \mathcal{E}\ e^{\underline{a}'\underline{U}} = \mathcal{E}\ e^{\underline{a}'_1\underline{U}_1}$ It should be clear that $\|\underline{a}\| = \|\underline{a}_1\|$.

Result 6. Let $\underline{U}_1, \underline{U}_2, \ldots, \underline{U}_n$ be independent and identically distributed random vectors with $\underline{U}_1 \sim U(S_m)$, and let $R_n = \left\|\sum_{i=1}^{n} \underline{U}_i\right\|$ then

(i) $\sum_{i=1}^{n} \underline{U}_i \sim R_n\underline{U}$,

(ii) $\underline{U} \sim U(S_m)$ and

(iii) R_n and $\underline{U}$ are independent,

Proof: See Mardia (1979), Chapter 15, section 15.4.

Result 7. With R_n as defined in Result 6, we have $R_n^2 = R_m^2 + R_{n-m}^2 + 2R_mR_{n-m}\cos\theta$ where R_m, R_{n-m} and $\cos\theta$ are mutually independent and $t = \cos\theta \sim W^{\nu}$.

Proof: Now

$$\sum_{i=1}^{n}\underline{U}_i = \sum_{i=1}^{m}\underline{U}_i + \sum_{i=m+1}^{n}\underline{U}_i \sim R_m\underline{U}_* + R_{n-m}\underline{U}_{**} + 2R_n\underline{R}_{n-m}\underline{U}'_*\underline{U}_{**},$$

where R_m is the length of $\sum_{i=1}^{m}\underline{U}_i$, R_{n-m} the length of $\sum_{i=m+1}^{n}\underline{U}_i$, $\underline{U}_*$ and $\underline{U}_{**}$ are unit vectors . Hence

$$R_n^2 = \Big(\sum_{i=1}^{n}\underline{U}_i\Big)'\Big(\sum_{i=1}^{n}\underline{U}_i\Big) \sim R_n^2 = R_m^2 + R_{n-m}^2 + 2\, R_mR_{n-m}\, \underline{U}'_{**}\underline{U}_*.$$

Since $\underline{U}_{**}$ is a unit vector then the density of t conditioning on $\underline{U}_{**}$, say $g(t|\underline{U}_{**})$, does not depend on $\underline{U}_{**}$ and by Result 2

$$g(t|\underline{U}_{**})\begin{cases} \dfrac{(1-t^2)^{\frac{1}{2}(m-3)}}{B(\frac{1}{2},\frac{1}{2}(m-1))} & -1<t<1 \\ 0 & \text{elsewhere.} \end{cases}$$

It follows that the unconditional density of t is the same as the conditional density, that is, $t \sim W^{\nu}$.

We conclude this chapter by summarizing a paper by Saw (1986).

1.4 On a Paper by Saw.

Formula (8.532) in Gradshteyn and Ryzhik(1980) us to write

$$\frac{J_{\nu}(tR_{m+n})}{R_{m+n}^{\nu}}=2^{\nu}t^{-\nu}\Gamma(\nu)\sum_{k=0}^{\infty}(\nu+k)\frac{J_{\nu+k}(tR_m)}{R_m^{\nu}}\ \frac{J_k(tR_n)}{R_n^{\nu}}\ C_k^{\nu}(\cos\ \theta)$$

where $R_{m+n}^2=R_m^2+R_n^2+2R_mR_n\cos\theta$; R_{n+m}, R_n, and R_m are the sides of a triangle such that the angle between R_m and R_n is θ, $C_n^{\nu}(\cos\theta)$ is Gegenbauer polynomial.

Again in Gradshteyn and Ryzhik(1980), formula (7.323.1), we have the result that

$$\int_0^{\pi} C_n^{\nu}(\cos\ \theta)\ \sin^{2\nu}\theta\ d\theta = \begin{cases} 0 & n=1,2,\dots \\ \dfrac{2^{-2\nu}\pi\Gamma(2\nu+1)}{\Gamma(\nu+1)^2} & n=0. \end{cases}$$

Noting the relationship between $\Lambda_{\nu}(t)$ and the Bessel function $J_{\nu}(t)$ and using the formulae mentioned above, Saw arrived at the apparently simple but powerful result

$$\mathcal{E}\,\Lambda_\nu(tR_n) = \left\{\Lambda_\nu(t)\right\}^n \qquad (1.4.1)$$

An alternative proof of (1.4.1) is to let $\underline{U}_i$ i=1,2,...,n be independent identically distributed random unit vectors then

$$\mathcal{E}\,e^{t\underline{\alpha}'(\underline{U}_1+\underline{U}_2+\dots+\underline{U}_n)} = \mathcal{E}\,e^{t(W_1+W_2+\dots+W_n)} = \left\{\Lambda_\nu(t)\right\}^n.$$

Now

$$(\underline{U}_1+\underline{U}_2+\dots+\underline{U}_n) = \|\underline{U}_1+\underline{U}_2+\dots+\underline{U}_n\|.\frac{(\underline{U}_1+\underline{U}_2+\dots+\underline{U}_n)}{\|\underline{U}_1+\underline{U}_2+\dots+\underline{U}_n\|},$$

and $\|\underline{U}_1+\underline{U}_2+\dots+\underline{U}_n\|$ is independent of $\frac{(\underline{U}_1+\underline{U}_2+\dots+\underline{U}_n)}{\|\underline{U}_1+\underline{U}_2+\dots+\underline{U}_n\|}$. Using the result of Theorem 1 we get

$$\mathcal{E}\,e^{\alpha'(\underline{U}_1+\underline{U}_2+\dots+\underline{U}_n)} = \mathcal{E}\,e^{tR_n\underline{\alpha}'\underline{U}} = \Lambda_\nu(tR_n).$$

The last equality is immediate on noting that $\underline{U}\sim U(S_m)$.

The usefulness of equation (1.4.1) lies in the fact that it gives a computational method for determining the distribution of R_n. We now develop the theory as given by Saw.

Let χ_f be a chi random variable with f degrees of freedom. It is straightforward to show that

$$\mathcal{E}\,\chi^{2k}_{f+2s} = 2^k\frac{\Gamma(k+\frac{1}{2}(f+2s))}{\Gamma(\frac{1}{2}(f+2s))},$$

and thus it should be apparent that

$$\mathcal{E}\ \Lambda_{\nu}(t\chi_{m+2s}) = \mathcal{E}\ \Lambda_{\frac{1}{2}m-1}(t\chi_{m+2s}) = \sum_{0}^{\infty}(\tfrac{1}{2}t^{2})^{k}\frac{\Gamma(\frac{1}{2}m)\Gamma(k+\frac{1}{2}(m+2s))}{k!\,\Gamma(\frac{1}{2}(m+2s))\Gamma(k+\frac{1}{2}m)}$$

$$= M(\tfrac{1}{2}m+s,\tfrac{1}{2}m:\tfrac{1}{2}t^{2}) = e^{\frac{1}{2}t^{2}}\ M(-s,\tfrac{1}{2}m:-\tfrac{1}{2}t^{2}) = e^{\theta}\ M(-s,\ \tfrac{m}{2}:-\theta)$$

where $\theta=\frac{1}{2}t^{2}$ and $M(a,b:z)$ is the confluent hypergeometric function. The fourth equality is an application of formula (9.212) in Gradshteyn and Ryzhik (1980).

A property of the confluent hypergeometric function $M(a,b:z)$ is that when a is a negative integer or when a equals zero the series has a finite number of terms. Thus

$$\mathcal{E}\ \Lambda_{\nu}(t\chi_{m}) = e^{\theta}.\ 1$$

$$\mathcal{E}\ \Lambda_{\nu}(t\chi_{m+2}) = e^{\theta}.(1+\theta/(\tfrac{1}{2}m))$$

$$\vdots$$

This can be written in matrix form as

$$\begin{bmatrix} \mathcal{E}\ \Lambda_{\nu}(t\chi_{m}) \\ \mathcal{E}\ \Lambda_{\nu}(t\chi_{m+2}) \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = e^{\theta}A\ D^{-1}\ \underline{\theta} = e^{\theta}\underline{m},$$

$$\text{where } A = \begin{bmatrix} 1 & 0 & 0 & . & . \\ 1 & 2 & 0 & . & . \\ 1 & 4 & 4 & 0 & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix},$$

$$D^{-1} = \begin{bmatrix} 1 & 0 & 0 & . & . \\ 0 & \frac{1}{m} & 0 & . & . \\ 0 & 0 & \frac{1}{m(m+2)} & . & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix},$$

$\underline{\theta}' = (1, \theta, \theta^2, \ldots)$ and $\underline{m} = A\ D^{-1}\underline{\theta}$

We can show by polynomial division that $\Lambda_\nu(t\sqrt{\frac{m}{n}})$ can be written as the product of $e^{\frac{1}{2}\frac{t^2}{n}}$ and $g(t^2,m,n)$ where $g(t^2,m,n)$ is a polynomial in t^2. Consequently we have

$$\mathcal{E}\ \Lambda_\nu(t\sqrt{\tfrac{m}{n}}\ R_n) = \left\{\Lambda_\nu(t\sqrt{\tfrac{m}{n}})\right\}^n = e^{\frac{1}{2}t^2}\ G(\tfrac{1}{2}t^2),$$

where $G(\frac{1}{2}t^2) = g(t^2,m,n)\Big\}^n$. We can express the polynomial $G(\frac{1}{2}t^2)$ as the dot product of two vectors, the k^th^ component of the first vector being the coefficient associated with

$(\frac{1}{2}t^2)^k$ and the k^{th} component of the second vector being the coefficient associated with $(\frac{1}{2}t^2)^k$, thus giving

$$\left\{\Lambda_\nu(t\sqrt{\tfrac{m}{n}})\right\}^n = e^\theta \; \underline{a}'\underline{\theta}.$$

Using the fact that $\underline{a}' = \underline{a}'DA^{-1}AD^{-1}$ we obtain the result

$$e^\theta\underline{a}'\underline{\theta} = e^\theta\underline{a}' \, D \, A^{-1} \, A \, D^{-1}\underline{\theta} = e^\theta\underline{a}' \, D \, A^{-1}\underline{m},$$

by equation (1.4.2).

Thus we can write $\mathcal{E}\,\Lambda_\nu(t\sqrt{\tfrac{m}{n}}\,R_n)$ as a linear combination of $\mathcal{E}\,\Lambda_\nu(t\chi_{m+2s})$ $s=0,1,2,\ldots$

Let E be the infinite diagonal matrix

$E = \text{diag}\,(1,2,4,8,\ldots)$

and $\underline{p}$ be the infinite vector

$\underline{p} = (P(\chi_m<rc), P(\chi_{m+2}<rc), P(\chi_{m+4}<rc), \ldots),$

and let B be the matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & . \\ -1 & 1 & 0 & 0 & . \\ 1 & -2 & 1 & 0 & . \\ . & . & . & . & . \\ . & . & . & . & . \end{bmatrix},$$

then $A^{-1} = E^{-1}B$, giving $\mathcal{E}\,\Lambda_\nu(t\sqrt{\tfrac{m}{n}}R_n) = e^\theta\underline{a}'DE^{-1}B\underline{m}$. From the result

$$\mathcal{E}\ \Lambda_\nu(t\sqrt{\tfrac{m}{n}}\ R_n) = e^{\theta}\underline{a}'\ D\ E^{-1}\ B\ \underline{m},$$

we conclude that

$$P(R_n<r) = \underline{a}'D\ E^{-1}\ B\ \underline{p}.$$

Thus Saw has demonstrated that for all $m\geq 2$ and all $n\geq 2$ we can write the distribution function of R_n as a linear combination of the distribution functions of chi random variables. This is an exact relationship involving an infinite sum. In Chapter II we will mention some of the problems one encounters in evaluating the distribution function of R_n using the method described above.

The usefulness of equation (1.4.1) is further demonstrated by considering the situation

$$R_n = \|d_1\underline{U}_1+d_2\underline{U}_2+\ldots+d_n\underline{U}_n\|^{\frac{1}{2}}$$

where d_i $i=1,2,\ldots,n$ are constants and $\underline{U}_i$ $i=1,2,..,n$ are independent random vectors with each having a uniform distribution on the unit m-sphere. Then from equation (1.4.1) we have

$$\mathcal{E}\ \Lambda_\nu(tcR_n) = \prod_{i=1}^{n}\Lambda_\nu(ctd_i) \qquad (1.4.3)$$

and by the same considerations we again obtain $\mathcal{E}\ \Lambda_\nu(tcR_n) = e^{\theta}\ \underline{a}'\underline{\theta}$ for some vector $\underline{a}$. It is now clear that we can write $\mathcal{E}\ \Lambda_\nu(tcR_n)$ as a linear combination of $\mathcal{E}\ \Lambda_\nu(t\chi_{m+2k})$ $k=0,1,2,\ldots$

Saw also discussed the situation in which the d_i's are independent and identically distributed with emphasis on a χ_{m+2s} distribution. Again by (1.4.1)

$$\mathcal{E}\ \Lambda_\nu\left(\frac{tR_n}{\sqrt{n}}\right) = \left\{\mathcal{E}\ \Lambda_\nu\left(\frac{t}{\sqrt{n}}\ \chi_{m+2s}\right\}^n\right.$$

$$=\{\ e^\theta M(-s,\ \nu+1;\ \tfrac{-\theta}{n})\}^n.$$

From our prior discussion $M(-s,\nu+1,\frac{-\theta}{n})$ is a polynomial of order s in θ. And so, our polynomial is now of finite degree.

CHAPTER II
A COMPUTATIONAL STUDY

2.1 On a Paper by Saw (Continued)

The result that

$$P\ (R_n<r) = \underline{a}'\ D\ E^{-1}\ B\ \underline{p},$$

gives us a method for computing the distribution function of R_n $n\geq 2$, for any $m\geq 2$, by using a linear combination of the distribution functions of some chi random variables. Furthermore, the structure of the matrices E^{-1} and B, and the relationship between $P(\chi_f<r)$ and $p(\chi_{f+2}<r)$ allow us to write and use the alternative expression

$$P(R_n<r) = \underline{a}'D\ \underline{\partial}\ P(\chi_m<r) \qquad (2.1.1)$$

where $\underline{\partial}$ is the infinite vector, $\underline{\partial} = (1,\partial,\partial^2,\ldots)$, and ∂ an operator defined by $\partial g(m) = \frac{1}{2}\left\{g(m+2) - g(m)\right\}$.

We can convey an idea of the ease of the use of equation (2.1.1) by examining how to obtain the vectors and matrices given in the equality. We saw that

$$\mathcal{E}\ \Lambda_\nu(tcR_n) = \left\{\Lambda_\nu(tc)\right\}^n = \Psi^n(\tfrac{\theta}{}\tfrac{m}{n}) = e^{\theta}\underline{a}'\theta.$$

where $\Psi(\theta)=\sum_{k=0}^{\infty}\theta^k\frac{1}{k!\ m(m+2)\ldots(m+2k-2)}$, $\theta=\frac{1}{2}t^2$ and $c=\sqrt{\frac{m}{n}}$.

The computation of the coefficient of θ^k in $\Psi^n(\frac{\theta}{n}\frac{m}{})$ is best effected by repeated use of an algorithm corresponding to polynomial multiplication. We can consider $\Psi(\frac{\theta}{n}\frac{m}{})$ as an infinite vector with the coefficient of θ^{j-1} as its j^{th} element j=1,2,... In order to obtain the coefficient of θ^k in the expansion of $\Psi^n(\frac{\theta}{n}\frac{m}{})$, we select the first k+1 elements of the vector $\Psi(\frac{\theta}{n}\frac{m}{})$, and on making n-1 applications of the algorithm, we obtain the first k+1 elements of the vector $\Psi^n(\frac{\theta}{n}\frac{m}{})$ with the last element being the coefficient of θ^k. The power series for $e^{-\theta}$ is known; we can now obtain the elements of $\underline{a}'$ by polynomial multiplication of $\Psi^n(\frac{\theta}{n}\frac{m}{})$ by $e^{-\theta}$ and letting the coefficient of θ^k k=0,1,2,... in the resulting polynomial be the k+1 element of $\underline{a}$.

The diagonal element, d_{kk}, of the infinite diagonal matrix D is known and can be written as $d_{kk}=m(m+2)(m+4)\ldots(m+2k-2)$ k=2,3,... and $d_{11}=1$.

The vectors and matrices we are considering each have an infinite number of elements. We must truncate these vectors and matrices to q+1 terms, q = 0,1,2,...; Saw refers to the value obtained as the q^{th} approximant to $P(R_n \leq r)$. When m=3, the density function of R_n is known in closed form. We can compare the values of the various approximants with exact distribution of R_n for different values of r and n. This is done in Table I, which is taken from Saw.

Result 2.1.1 is exact. However, in order to use this result we must truncate the vectors and matrices given in the expression. Table II indicates that the approximants do not rapidly converge to the exact value of the distribution function of R_n when n is small ($n \leq 9$).

2.2 A Computational Study

Examine Table II of section 2.1 and note that for m=10, n=2 and r=0·5, the 10th approximant is equal to -0.0001. We obtain negative values for some approximants since some of the coefficients, in the linear combination of the distribution functions of chi random variables, are negative.

A chi random variable is defined on the positive half of the real line, while R_n is defined on a finite interval. The slow convergence of the approximants towards the exact value indicates that we must include a large number of terms in order to have the linear combination of the distribution functions of chi random variables be a good approximation for the distribution function of R_n.

A part of our research was to obtain a set of random variables $\left\{Y_k\right\}_{k=1}^{\infty}$ such that

$$\mathcal{E}\ \Lambda_\nu(tR_n) = a_o\{\Lambda_\nu(t)\}^2 + \sum_{k=1}^{\infty} a_k \mathcal{E}\ \Lambda_\nu(t\ Y_k)$$

TABLE I

Tabulated is

diff=$|P\{R_n{}^{(3)}\leq r\} - P_q\{R_n{}^{(3)}\leq r\}|$ and reldiff = diff/$P\{R_n{}^{(3)}\leq r\}$

for some specimen values of n, r and q.

n = 3	q = 0		q = 10	
	diff	reldiff	diff	reldiff
r = 0.5	1.0 E-2	4.8 E-1	7.0 E-5	3.3 E-3
r = 1.0	3.2 E-2	1.9 E-1	6.9 E-4	4.2 E-3
r = 1.5	4.0 E-2	9.2 E-2	9.5 E-4	2.2 E-3
r = 2.0	3.0 E-2	4.3 E	7.4 E-4	1.0 E-3
r = 2.5	1.7 E-2	1.8 E-2	1.5 E-3	1.6 E-3
r = 30.	2.9 E-2	2.9 E-2	2.1 E-3	2.1 E-3

n = 4	q = 0		q = 5	
	diff	reldiff	diff	reldiff
r = 0.5	2.5 E-3	1.4 E-1	4.9 E-4	2.7 E-2
r = 1.5	3.5 E-2	1.1 E-1	9.8 E-4	3.0 E-3
r = 2.5	6.0 E-3	7.5 E-3	1.9 E-3	2.4 E-3
r = 3.5	1.7 E-2	1.8 E-2	2.3 E-3	2.3 E-3

n = 6	q = 0		a = 5	
	diff	reldiff	diff	reldiff
r = 0.5	1.3 E-3	1.3 E-1	5.2 E-5	5.2 E-3
r = 2.0	2.2 E-2	5.4 E-2	4.3 E-4	1.1 E-3
r = 3.5	6.3 E-3	6.9 E-3	2.6 E-4	2.9 E-4
r = 5.0	4.5 E-3	4.5 E-3	1.6 E-4	1.6 E-4

TABLE II

Comparison of various approximants with the exact value

n = 2 r = 0.50

dim =	10	20	30	40	50	Exact
m = 2	0.1742	0.1621	0.1732	0.1767	1.1685	0.1609
m = 3	0.0681	0.0644	0.0667	0.0633	0.0621	0.0625
m = 4	0.0268	0.0277	0.0271	0.0257	0.0267	0.0260
m = 5	0.0104	0.0120	0.0112	0.0117	0.0110	0.0112
m = 6	0.0039	0.0051	0.0049	0.0051	0.0049	0.0050
m = 7	0.0013	0.0021	0.0022	0.0022	0.0023	0.0022
m = 8	0.0003	0.0009	0.0010	0.0010	0.0010	0.0010
m = 9	0.0000	0.0004	0.0005	0.0004	0.0005	0.0005
m = 10	-0.0001	0.0002	0.0002	0.0002	0.0002	0.0002

n = 2 r = 1.00

dim=	10	20	30	40	50	Exact
m = 2	0.3224	0.3384	0.3386	0.3294	0.3337	0.3333
m = 3	0.2401	0.2518	0.2483	0.2491	0.2491	0.2500
m = 4	0.1915	0.1989	0.1940	0.1942	0.1964	0.1955
m = 5	0.1561	0.1582	0.1554	0.1569	0.1568	0.1562
m = 6	0.1283	0.1267	0.1257	0.1270	0.1268	0.1266
m = 7	0.1060	0.1028	0.1031	0.1037	0.1038	0.1035
m = 8	0.0880	0.0844	0.0853	0.0854	0.0851	0.0852
m = 9	0.0732	0.0698	0.0708	0.0706	0.0705	0.0706
m = 10	0.0612	0.0581	0.0589	0.0586	0.0586	0.0587

Table II(continued)

n=2 r=1.4142

dim=	10	20	30	40	50	Exact
m = 2	0.4837	0.4769	0.4856	0.4934	0.4982	0.5000
m = 3	0.4989	0.4928	0.4944	0.5006	0.5038	0.5000
m = 4	0.5020	0.4922	0.4964	0.5012	0.5024	0.5000
m = 5	0.5012	0.4951	0.4993	0.5013	0.5013	0.5000
m = 6	0.5005	0.4972	0.4995	0.5007	0.5007	0.5000
m = 7	0.5005	0.4977	0.4995	0.5005	0.5004	0.5000
m = 8	0.5009	0.4979	0.4999	0.5005	0.5003	0.5000
m = 9	0.5015	0.4983	0.5000	0.5005	0.5003	0.5000
m = 10	0.5020	0.4987	0.4999	0.5002	0.5001	0.5000

where $a_k \geq 0$ and $\sum_{k=1}^{\infty} a_k = 1$. We must report failure in this area of the research but we now outline some of our methods.

For n=2 and any $m \geq 2$, it is known that $\left(\frac{1}{2}R_n\right)^2$ has a $B\left(\frac{1}{2}(m-1), \frac{1}{2}(m-1)\right)$ distribution, thus an intuitive approach is to conduct the search amongst the family of beta random variables.

Furthermore $\frac{1}{2}(m-1) = \frac{1}{2}(m+j-3)$ where j=2 leads us to consider the set of beta random variables

$$\left\{B\left(\tfrac{1}{2}(m-1),\ \tfrac{1}{2}(m+j-3)\right)\right\}_{j=2}^{\infty}.$$

A second approach was based on examining the first two moments of $\left(\frac{R_n}{n}\right)^2$. These moments can be obtained by a simple manipulation of the function $\Lambda_\nu\left(\frac{t}{n}\right)$. Next we selected a beta random variable; its first two moments agree with the first two moments of $\left(\frac{R_n}{n}\right)^2$. Such a random variable has a $B\left(\frac{1}{2}(mn-2), \frac{1}{2}(mn-2)(n-1)\right)$ distribution. Thus a second set of random variables for which we could investigate the relationship was

$$\left\{ B\left(\frac{1}{2j}(mj-2),\ \frac{1}{2j}(mj-2)(j-1)\right)\right\} \qquad j=2,3,4,\ldots$$

The method of selecting a beta random variable by matching the first two moments appears in a paper by N.L. Johnson (1965).

Table III gives the coefficient of $(\frac{1}{4}t^2)^k$, k=0,1,...,6, in the expansion of $\left\{\Lambda_\nu(\frac{t}{n})\right\}^n$ when $\nu=\frac{1}{2}$ and n=2,3,...,8. Table IV gives the coefficient of $(\frac{1}{4}t^2)^k$ k=0,1,...,6, in the expansion of

$$\mathcal{E}\ \Lambda_\nu(t\ B\left(\frac{(mj-2)}{2j},\ \frac{(mj-2)(j-1)}{2j}\right)\ )$$

where $\nu=\frac{1}{2}$ and j=2,3,4,...,8. Note that the row corresponding to n=2 and that the first two columns of Table III are the first two columns of Table IV.

The data in Table III and Table IV can be considered as the elements of two matrices, each having seven rows and seven columns. We can now obtain the coefficients a_k k=0,1,2,...,6, for a fixed value of n by postmultiplying the inverse of the transpose of the matrix in Table IV by the transpose the appropriate row from Table III. The coefficients obtained from solving this linear system were negative at times; and so we did not continue with this method of examining the moments of sets of beta random variables.

2.3 Conclusion

Our investigations were more extensive than indicated above. The program which gave the output for the data of Table III and Table IV with a few minor changes can be used to examine higher dimensions and different sets of Beta random variables. This was done. In higher dimensions the results were more appealing but coefficients (a_k, k=0,1,2,..), were at times, negative and their absolute values could be larger than unity.

Our failure to find a suitable set of random variables does not indicate that the problem does not have a solution but rather that we must develop a better methodology. Our methods were not much more than educated guesswork; perhaps there are functional relationships we could have exploited but which we were not aware of. The fact that the linear combination of distribution functions of chi random variables can be replaced by a single and easily computed distribution function, when n=2, leads us to believe that further investigations of this problem may be profitable. A next method of attack is needed in order to make further progress towards the solution of this problem.

TABLE III

Coefficient $(\frac{t^2}{2})^k$ in the expansion of $\{\Lambda_\nu(\frac{t}{\sqrt{n}})\}^n$

m=3

	k=2	k=3	k=4	k=5
n=2	1.0000000000	0.3333333333	0.0444444444	0.0031746031
n=3	1.0000000000	0.2222222222	0.0213991769	0.0011902947
n=4	1.0000000000	0.1666666666	0.0125000000	0.0005621693
n=5	1.0000000000	0.1333333333	0.0081777777	0.0003074708
n=6	1.0000000000	0.1111111111	0.0057613168	0.0001206350
n=7	1.0000000000	0.0952380952	0.0042759961	0.0001206350
n=8	1.0000000000	0.8333333333	0.0032986111	0.0000826719

	k=6	k=7	k=8
n=2	0.0001410934	0.0000042755	0.00000000939
n=3	0.0000432893	0.0000011099	0.00000000211
n=4	0.0000170855	0.0000003757	0.00000000062
n=5	0.0000079770	0.0000001525	0.00000000022
n=6	0.0000041934	0.0000000707	0.00000000009
n=7	0.0000024059	0.0000000361	0.00000000004
n=8	0.0000014758	0.0000000200	0.00000000002

TABLE IV

Coefficient $(\frac{t^2}{2})^k$ in the expansion of

$$\mathcal{E}\ \{\Lambda_\nu(t\ B(\frac{mj-2}{2j}, \frac{(mj-2)(j-1)}{2j})^{\frac{1}{2}}\}$$

m=3

	k=2	k=3	k=4	k=5
j=2	1.0000000000	0.3333333333	0.0444444444	0.0031746031
j=3	1.0000000000	0.2222222222	0.0213991771	0.0011734036
j=4	1.0000000000	0.1666666666	0.0125000000	0.0005527210
j=5	1.0000000000	0.1333333333	0.0081777778	0.0003023716
j=6	1.0000000000	0.1111111111	0.0057613173	0.0001828989
j=7	1.0000000000	0.0952380880	0.0042759955	0.0001188830
j=8	1.0000000000	0.0833333333	0.0032986111	0.0000815590

	k=6	k=7	k=8
j=2	0.0001410934	0.0000042755	0.00000000939
j=3	0.0000417878	0.0000010468	0.00000001947
j=4	0.0000163129	0.0000003460	0.00000000055
j=5	0.0000076034	0.0000001395	0.00000000019
j=6	0.0000040028	0.0000000646	0.00000000008
j=7	0.0000023021	0.0000000332	0.00000000003
j=8	0.0000014159	0.0000000184	0.00000000001

CHAPTER III
ON THE DISTRIBUTION OF R_n

3.1 Literature Review

Determination of the probability density of R_n, the length of the sum of n independent random vectors, is a problem which has been investigated by a considerable number of workers. Rayleigh (1905) gave an approximate solution in the distribution of the resultant when each of the independent random vectors has a uniform distribution on the circle. At this time his work was motivated by a question posed by Pearson (1905). At a later period, Rayleigh gave an exact solution to the three dimensional case. Then his investigations were more extensive as he also examined the probability density of R_n in one and two dimensions.

Kluvyer (1905) gave an exact solution to the two dimensional case. However, his answer is in the form of an integral, with the integral being a product of Bessel functions. Evaluation of this integral, even with the use of numerical techniques, is difficult. G.N. Watson (1944), on page 421, in his treatise on the Theory of Bessel Functions extends Kluvyer's method to an arbitrary number of dimensions; Watson's solution is given by

$P(R_n < r | a_1, \ldots, a_n) =$

$$r\left\{\Gamma(\tfrac{1}{2}m)\right\}^{n-1}\int_0^\infty (\tfrac{1}{2}rt)^{\frac{1}{2}m-1} J_{\frac{1}{2}m}(rt)\prod_{n=1}^{n}\left\{ \frac{J_{\frac{1}{2}m-1}(a_i t)}{(\frac{1}{2}a_i t)^{\frac{1}{2}m-1}} \right\}dt \qquad (3.1.1)$$

where a_i: $i=1,2,\ldots,n$ is the length of the i^{th} random vector.

Like Kluvyer's solution the right side of equation (3.1.1) is seen to be an integral which involves the product of Bessel functions. Even when computed by the use of numerical integrations methods, convergence of this integral is slow; thus, an alternative method of computing the distribution function of R_n is of some importance to workers in this area.

In this chapter we develop such a method. We first introduce the situations where an alternative method is already known in section 3·2 and section 3·3. In section 3.2 we give the distribution of R_2 for any $m \geq 2$ whereas in section (3·3) we discuss the distribution of R_n ($m=3$). In section (3.4) we give the distribution of R_n for odd values of m.

3.2 On the Distribution of R_n

In Chapter I we mentioned the distribution of $\underline{\alpha}'\underline{U}$, where $\underline{\alpha}$ is a unit vector and $\underline{U}$ has a uniform distribution on the m-sphere.

It is this result which allows us to derive the probability density function of R_2 for $m\geq 2$.

Theorem 1. Let $\underline{U}_1$, and $\underline{U}_2$ be independent identically distributed random vectors with $\underline{U}_1 \sim U(S_m)$. Let a_1, and a_2 be positive real numbers and R_2 be the resultant of $a_1\underline{U}_1+a_2\underline{U}_2$; then the probability density function of R_2 is given by

$$h(r) = \begin{cases} \dfrac{r}{a_1a_2B(\frac{1}{2},\frac{1}{2}(m-1))} \left[\dfrac{(r^2-(a_1-a_2)^2)((a_1+a_2)^2-r^2)}{2a_1a_2}\right]^{\frac{1}{2}(m-3)} & |a_1-a_2|<r<(a_1+a_2) \\ 0 & \text{elsewhere.} \end{cases}$$

Proof: The scalar product $(a_1\underline{U}_1 + a_2\underline{U}_2)'(a_1\underline{U}_1 + a_2\underline{U}_2)$ gives

$$R_2^2 = a_1^2 + a_2^2 + 2a_1a_2\cos\theta,$$

where θ is the angle between $\underline{U}_1$ and $\underline{U}_2$. We saw that

$T = \cos\theta \sim W^{\nu}$.

The quantity $a_1^2 + a_2^2 + 2a_1a_2\cos\theta$ attains its maximum value when $\theta=0$ and its minimum value when $\theta=\pi$.

Thus for $|a_1-a_2|< r <(a_1+a_2)$

$$P(R_2 <r) = P(R_2^2 < r^2) = P\left(\cos\theta < \frac{r^2-a_1^2-a_2^2}{2a_1a_2}\right),$$

and on denoting $\frac{r^2-a_1^2-a_2^2}{2a_1a_2}$ by $m(r)$ we get

$$P\left(\cos\theta < m(r)\right) = \int_{-1}^{m(r)} \frac{(1-x^2)^{\frac{1}{2}(m-3)}}{B(\frac{1}{2},\frac{1}{2}(m-1))}dx.$$

Denoting the distribution function of R_2 by $H(r)$ we write

$$H(r) = \begin{cases} 0 & r < |a_1-a_2| \\ P\left(\cos\theta < m(r)\right) & |a_1-a_2| < r < (a_1+a_2) \\ 1 & r > (a_1+a_2). \end{cases}$$

As a result of the Liebenitz theorem for differentiating under the integral sign, we obtain the density of R_2 as

$$h(r) = \begin{cases} \frac{r}{a_1a_2} \frac{[1-(\frac{r^2-a_1^2-a_2^2}{2a_1a_2})^2]^{\frac{1}{2}(m-3)}}{B(\frac{1}{2},\frac{1}{2}(m-1))} & |a_1-a_2| < r < (a_1+a_2) \\ 0 & \text{elsewhere,} \end{cases}$$

and on noting that the expression in square brackets is a difference of two squares we write $h(r)$ as

$$h(r) = \frac{r}{a_1a_2B(\frac{1}{2},\frac{1}{2}(m-1))} \cdot \left[\frac{(r^2-(a_1-a_2)^2)((a_1+a_2)^2-r^2)}{2a_1a_2}\right]^{\frac{1}{2}(m-3)}.$$

Letting $a_1 = a_2 = 1$ gives density function of the length of the resultant of the sum of two independent random vectors on the unit sphere as

$$h(r) = \frac{r}{B(\frac{1}{2},\frac{1}{2}(m-1))}[\tfrac{1}{2}r^2(4-r^2)]^{\frac{1}{2}(m-3)} \qquad 0 < r < 2.$$

We now develop the three dimensional case.

3.3 The Three Dimensional Problem

In section 1·4 we saw that

$$\mathcal{E}\,\Lambda_\nu(tR_n) = \left\{\Lambda_\nu(t)\right\}^n \tag{3.3.1}$$

where $\nu = \frac{1}{2}m-1$ and m denotes the dimensionality of the problem. We can obtain a similar result with the function $V_\nu(t)$, where $V_\nu(t) = \Lambda_\nu(it)$, thus

$$\mathcal{E}\,V_\nu(tR_n) = \left\{V_\nu(t)\right\}^n \tag{3.3.2}$$

Let X be a random variable having a uniform distribution on the interval [-1,1] then the characteristic function of X, $\mathcal{E}\,e^{itx}$, is given by

$$\mathcal{E}\,e^{itx} = \frac{e^{it}-e^{-it}}{2it} = V_{\frac{1}{2}}(t) \tag{3.3.3}$$

Equation (3.3.2) and equation (3.3.3) are useful in deriving the probability density and probability distribution function of R_n for the three dimensional case.

Theorem 2. Let Y be a random variable with probability density function, g(y), satisfying

(i) $g(y) = g(-y)$,

$$(ii) \quad g^{(1)}(y) = \begin{cases} >0 & y < 0 \\ =0 & y = 0 \\ <0 & y < 0. \end{cases}$$

Let $\phi_Y(t) = \mathcal{E}\, e^{ity}$ be the characteristic function of Y. Then $-2rg^{(1)}(r)$ is the probability density function of a positive random variable say R, such that

$$\mathcal{E}\, V_{\frac{1}{2}}(tR) = \phi_Y(t).$$

Proof: Clearly $-2rg^{(1)}(r) > 0$, and

$$\int_0^\infty -2rg^{(1)}(r)dr = -2r\, g(r)\,|_0^\infty + 2\int_0^\infty g(r)dr = 1,$$

hence $-2rg^{(1)}(r)$, $r>0$, is a probability density function. Also

$$\mathcal{E}\, V_{\frac{1}{2}}(tr) = \int_0^\infty \frac{e^{itr}-e^{-itr}}{2itr}\, [-2rg^{(1)}(r)]dr$$

$$= \int_0^\infty \left(\frac{e^{it}-e^{-itr}}{it}\right) g^{(1)}(r)\, dr$$

$$= \left(\frac{e^{itr}-e^{-itr}}{it}\right) g(r)\ |_0^\infty + \int_0^\infty (e^{itr}-\ e^{-itr})\ g(r)\ dr$$

$$= \int_{-\infty}^\infty e^{itr}\ g(r)\ dr = \phi_Y(t),$$

which is the result we wanted to establish.

Corollary. If we denote the probability distribution function of R by H(r) and that of Y by G(y) then

$$H(r) = 2G(r) - 1 - 2rg(r).$$

Proof: We know that

$$H(r) = P(R < r) = -\int_0^r 2t\ g^{(1)}(t)\ dt$$

$$= -2t\ g(t)\ \Big|_0^r + 2\int_0^r g(t)\ dt$$

$$= -2r\ g(r) + 2\ G(r) - 2G(0)$$

$$= 2G(r) - 1 - 2rg(r)$$

which is the required result.

By letting $\overline{G}(r) = 1 - G(r)$ and $\overline{H}(r) = 1 - H(r)$ we get the alternative expression

$$\tfrac{1}{2}\overline{H}(r) = \overline{G}(r) + r\ g(r).$$

Let $Y = \sum_{i=1}^{n} X_i$ where X_i $i=1,2,\ldots,n$ are independent identically distributed random variables with the probability distribution function of X_1 given by

$$f(x_1) = \begin{cases} \frac{1}{2} & -1 \leq x_1 \leq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Then, from the discussion preceding Theorem 2, we have the result

$$\phi_Y(t) = \left\{V_{\frac{1}{2}}(t)\right\}^n = \mathcal{E}\ V_{\frac{1}{2}}(tR_n).$$

Theorem 2 and the corollary following this theorem give us the distribution of R_n for the three dimensional case we have

$$h(r) = -2r\ g^{(1)}(r) \qquad\qquad r>0,$$

and

$$\tfrac{1}{2}\bar{H}(r) = \bar{G}(r) + r\ g(r),$$

where $h(r)$ and $H(r)$ are the probability density function and probability distribution function of R_n. The functions $g(r)$ and $G(r)$ are the density function and distribution respectively, of the random variable Y.

We conclude this section by giving the density and distribution function of Y. A derivation can be found in the works of Feller (1971), Cramer (1946) and Rao (1973) amongst others.

The density function $g(y)$ is given by

$$g(y) = \frac{1}{2^n(n-1)!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} [y+n-2j]^{n-1} \qquad (3.3.4)$$

and distribution function by

$$G(y) = \frac{1}{2^n n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} [y+n-2j]^n \qquad (3.3.5)$$

where $-n \leq y \leq n$ and

$$[y+n-2j] = \begin{cases} 0 & (y+n-2j) < 0 \\ (y+n-2j) & (y+n-2j) \geq 0. \end{cases}$$

3.4 Higher Dimension (m, odd).

The literature on the subject of Bessel functions is quite rich. These functions have been studied in depth and their properties well tabulated; see for example Watson (1944). We mentioned the relationship between $V_n(t)$ and the Bessel function $J_n(t)$ in Chapter I. We now show that $V_\nu(t)$ like $J_\nu(t)$, satisfies a differential difference equation. It is this property of $V_\nu(t)$ which allows us to develop the results of this section.

Lemma. The function $V_\nu(t)$ satisfies the differential difference equation $\frac{d}{dw}\left\{w^{2\nu}V_\nu(tw)\right\} = 2\nu w^{2\nu-1}V_{\nu-1}(tw)$ (3.4.1)

Proof. The series representation of $w^{2\nu}V_\nu(tw)$ is

$$w^{2\nu}V_\nu(tw) = \sum_{k=0}^{\infty} \frac{\Gamma(\nu+1)}{k!\,\Gamma(\nu+k+1)} (-1)^k (\tfrac{1}{2}t)^{2k}\, w^{2\nu+2k}.$$

Since the series is uniformly convergent we can interchange the operations of summation and differentiation to obtain

$$\frac{d}{dw}\left\{w^{2\nu}V_\nu(tw)\right\} = \sum_{k=0}^{\infty} \frac{\Gamma(\nu+1)(2\nu+2k)}{k!\Gamma(\nu+k+1)}(-1)^k(\tfrac{1}{2}t)^{2k}\; w^{2\nu+2k-1},$$

which on simplifying gives

$$\frac{d}{dw}\left\{w^{2\nu}V_\nu(tw)\right\} = 2\nu \sum_{k=o}^{\infty} \frac{\Gamma(\nu)}{k!\Gamma(\nu+k)}(-1)^k(\tfrac{1}{2}t)^{2k}\; w^{2\nu+2k-1}$$

$$= 2\nu\; w^{2\nu-1}V_{\nu-1}(tw),$$

which is the result we wanted to prove.

Multiplying both sides of the equality in equation (3.4.1) by w^{-1} we get

$$\left(\frac{d}{wdw}\right)w^{2\nu}V_\nu(tw) = 2\nu w^{2\nu-2}V_{\nu-1}(tw) \qquad (3.4.2)$$

which leads us to conclude that

$$\left(\frac{d}{wdw}\right)^k\{w^{2\nu}V_\nu(tw)\} = 2^k\nu(\nu-1)\ldots(\nu-k+1)w^{2\nu-2k}\; V_{\nu-k}(tw) \qquad (3.4.3)$$

Let W be a random variable with a symmetric probability density functions, say g(w), then the characteristic function of W can be written as

$$\mathcal{E}\; e^{itW} = \int_{-\infty}^{\infty} e^{itw}g(w)dw = \int_{0}^{\infty}\left(\frac{e^{itw}+e^{-itw}}{2}\right)g(w)\; dw$$

$$= \int_0^\infty V_{-\frac{1}{2}}(tw)\ g(w)\ dw.$$

Furthermore, if $W=\sum_{i=1}^{n} W_i$, where the W_i's are independent identically distributed random variables with $W_1 \sim W^\nu$, then

$$\mathcal{E}\ e^{itW} = \int_0^\infty V_{-\frac{1}{2}}(tw) g(w) dw$$

$$= [\int_{-1}^{1} e^{itx} \frac{(1-x^2)^{\nu-\frac{1}{2}}}{B(\frac{1}{2},\ \nu+\frac{1}{2})}\ dx]^n$$

$$= [V_\nu(t)]^n,$$

and on denoting the density of R_n by $h(r)$, we have the result

$$\int_0^\infty V_\nu\ (tr)h(r)dr = \left\{V_\nu(t)\right\}^n = \int_0^\infty V_{-\frac{1}{2}}(tw)g(w)dw \qquad (3.4.4)$$

Equations (3.4.1) through (3.4.4) will be useful in determining $h(r)$, the probability density function of R_n, from a knowledge of the density of $\sum_{i=1}^{n} W_i$ with $W_i \sim W^\nu$.

In this section the number of dimension m is related to the variable k by the equality $m = 2k+1 = 3,5,7,\ldots$

We now develop our main result by the principle of mathematical induction.

Theorem 3. Let $g_k(x)$ $k=1,2,\ldots$ be a sequence of functions such that the first k derivatives of $g_k(x)$ exist and let the functions

$$V_{\frac{1}{2}(2j-1)}(tx)x^{2j}\left[\left(\frac{d}{xdx}\right)^j g_k(x)\right], \qquad j=0,1,2,\ldots,k$$

be integrable, then

$$\int_{-\infty}^{\infty} V_{-\frac{1}{2}}(tx)\,g_k(x)\,dx=$$

$$\frac{(-1)^k}{1.3\ldots(2k-1)}\int_{-\infty}^{\infty} V_{\frac{1}{2}(2k-1)}(tx)x^{2k}\left[\left(\frac{d}{xdx}\right)^k g_k(x)\right]dx \tag{3.4.5}$$

Proof: Observe that

$$\int_{-\infty}^{\infty} V_{-\frac{1}{2}}(tx)\ g_1(x)\ dx = \int_{-\infty}^{\infty} \frac{d}{dx}[x\ V_{\frac{1}{2}}(tx)]\ g_1(x)\ dx$$

$$= x\ V_{\frac{1}{2}}(tx)\ g_1(x)\ \Big|_{-\infty}^{\infty} \qquad - \int_{-\infty}^{\infty} x\ V_{\frac{1}{2}}(tx)\ g_1^{(1)}(x)dx$$

$$= -\int_{-\infty}^{\infty} V_{\frac{1}{2}}(tx)\ x^2\left[\left(\frac{d}{xdx}\right)g_1(x)\right]\ dx.$$

Thus the result is true for $k=1$. (Note that the function $xg_1^{(1)}(x)$ is similar to the density of R_n in section 3.3.)

Assume that the result is true for $k=i$, that is

$$\int_{-\infty}^{\infty} V_{\frac{1}{2}}(tx)g_i(x)dx = \frac{(-1)^i}{1.3.5\ldots(2i-1)}\int_{-\infty}^{\infty} V_{\frac{1}{2}(2i-1)}(tx)x^{2i}\ \left[\left(\frac{d}{xdx}\right)^i g_i(x)\right]dx.$$

It is clear from equation (3.4.1) that

$$\frac{d}{dx}(x^{2i+1}\ V_{\frac{1}{2}(2i+1)}(tx)) = (2i+1)\ x^{2i}\ V_{\frac{1}{2}(2i-1)}(tx)$$

thus for k= i+1 we can write

$$\int_{-\infty}^{\infty} V_{-\frac{1}{2}}(tx)\, g_{i+1}(x)\, dx = \frac{(-1)^{i}}{1.3...(2i-1)} \int_{-\infty}^{\infty} V_{\frac{1}{2}(2i-1)}(tx)\ \ x^{2i}[(\frac{d}{xdx})^{i} g_{i+1}(x)]\,dx$$

$$= \frac{(-1)^{i}}{1.3...(2i+1)} x^{2i+1} V_{\frac{1}{2}(2i+1)}(tx)[(\frac{d}{xdx})^{i} g_{i+1}(x)]\ +$$

$$\frac{(-1)^{i+1}}{1.3.5...(2i+1)} \int_{-\infty}^{\infty} V_{\frac{1}{2}(2i+1)}(tx)\ \ x^{2i+2}[(\frac{d}{xdx})^{i+1} g_{i+1}(x)]\,dx$$

$$= \frac{(-1)^{i+1}}{1.3.5...(2i+1)} \int_{-\infty}^{\infty} V_{\frac{1}{2}(2i+1)}(2tx)\, x^{2i+2}[(\frac{d}{xdx})^{i+1} g_{i+1}(x)]\,dx$$

which is the result we set out to prove; thus, by the principle of mathematical induction, the result is true for all positive integers k.

Corollary: Let $g_k(x)$ be a sequence of even functions that satisfies the conditions of Theorem 3, then

$$\int_{-\infty}^{\infty} V_{-\frac{1}{2}}(tx)\, g_k(x)\, dx = 2\int_{0}^{\infty} V_{-\frac{1}{2}}(tx)\, g_k(x)\, dx$$

$$= \frac{2(-1)^{k}}{1.3...(2k-1)} \int_{0}^{\infty} V_{\frac{1}{2}(2k-1)} x^{2k}[(\frac{d}{xdx})^{k} g_k(x)]dx$$

Proof: Now, $V_{\frac{1}{2}(2k-1)}(tx)x^{2k}[(\frac{d}{xdx})^{k} g_k(x)]$ is an even function since it is the product of even functions.

The second integral of equation (3.4.5) may now be written as

$$\frac{(-1)^k}{1.3...(2k-1)} \int_{-\infty}^{\infty} V_{\frac{1}{2}(2k-1)}(tx)\ x^{2k}[(\frac{d}{xdx})^k g_k(x)]\ dx$$

$$= \frac{2(-1)^k}{1.3...(2k-1)} \int_{0}^{\infty} V_{\frac{1}{2}(2k-1)}(tx)\ x^{2k}[(\frac{d}{xdx})^k g_k(x)]\ dx$$

which completes the proof.

Corollary: Let $W_i \sim W^{\nu}$ $i = 1,2,...,n$ be independent random variables. Let $g(w)$ be the probability density function of $\sum_{i=1}^{n} W_i$ and $h(w)$ denote the probability density function of R_n in the m-dimensional case, then

$$h(w) = \frac{2(-1)^{\frac{1}{2}(m-1)}}{1.3.5...(m-2)}\ w^{m-1}\left(\frac{d}{wdw}\right)^{\frac{1}{2}(m-1)} g(w)$$

Proof: First define the even function $h^*(w)$ by

$$h^*(w)=h^*(-w)=\frac{h(w)}{2}, \qquad w \geq o.$$

From the discussion preceding Theorem 2, we know that

$$\int_{-\infty}^{\infty} V_{\frac{1}{2}m-1}(tw)h^*(w)dw = \int_{0}^{\infty} V_{\frac{1}{2}m-1}(tw)h(w)dw = \int_{-\infty}^{\infty} V_{-\frac{1}{2}}(tw)g(w)dw.$$

As a result of Theorem 3, we write this equality as

$$\int_{-\infty}^{\infty} V_{\frac{1}{2}m-1}(tw)\,h^*(w)\,dw=\int_{-\infty}^{\infty} V_{\frac{1}{2}m-1}(tw)\,\frac{(-w^2)^{\frac{1}{2}(m-1)}}{1.3.5...(m-2)}[(\frac{d}{wdw})^{\frac{1}{2}(m-1)}g(w)]\,dw$$

which can be written as

$$\int_{-\infty}^{\infty} V_{\frac{1}{2}m-1}(tw)\,[h^*(w)-\frac{(-w^2)^{\frac{1}{2}(m-1)}}{1.3.5...(m-2)}(\frac{d}{wdw})^{\frac{1}{2}(m-1)}g(w)]\,dw=0 \qquad (3.4.6)$$

We know that the integrand in equation (3.4.6) is a continuous function in the variables t and w. Multiplying this equation by $t^{2\nu}$ we can then differentiate both sides of this equality with respect to t. Equation (3.4.3) is indicative of how we must proceed from here. We get

$$0 = (\frac{d}{tdt})^{\frac{1}{2}(m-1)}\int_{-\infty}^{\infty} t^{2\nu}V_{\frac{1}{2}m-1}(tw)\Big(h^*(w) - \frac{(-w^2)^{\frac{1}{2}(m-1)}}{1.3...(m-2)}[(\frac{d}{wdw})^{\frac{1}{2}(m-1)}g(w)]\Big)\,dw.$$

Since the functions $h^*(w)$ and $\frac{(-w^2)^{\frac{1}{2}(m-1)}}{1.3...(m-2)}\Big\{(\frac{d}{wdw})^{\frac{1}{2}(m-1)}g(w)\Big\}$ are even functions, it is possible to write the equality above as

$$\int_{0}^{\infty} e^{itw}[h^*(w) - \frac{(-w^2)^{\frac{1}{2}(m-1)}}{1.3...(m-2)}\Big\{(\frac{d}{wdw})^{\frac{1}{2}(m-1)}g(w)\Big\}]\,dw = 0$$

which leads us to conclude that

$$h^*(w) = \frac{(-w^2)^{\frac{1}{2}(m-1)}}{1.3...(m-2)}\Big\{(\frac{d}{wdw})^{\frac{1}{2}(m-1)}g(w)\Big\}$$

for all values of w.

In particular for $w \geq 0$ we get

$$h(w) = 2\,\frac{(-w^2)^{\frac{1}{2}(m-1)}}{1.3\ldots(m-2)}\left\{\left(\frac{d}{wdw}\right)^{\frac{1}{2}(m-1)} g(w)\right\} \qquad (3.4.7)$$

which is the result we set out to prove.

In our proof of the corollaries following Theorem 3, we used the result that $g(w)$ is an even function in order to show that

$$h(w) = 2\,\frac{(-w^2)^{\frac{1}{2}(m-1)}}{1.3\ldots(m-2)}\left\{\left(\frac{d}{wdw}\right)^{\frac{1}{2}(m-1)} g(w)\right\} \quad m \geq 3.$$

This is important. The function $V_{-\frac{1}{2}}(tw)$ is an even function; therefore, when $g(w)$ is an even function we get

$$\int_{-\infty}^{\infty} V_{-\frac{1}{2}}(tw)g(w)dw = 2\int_{0}^{\infty} V_{-\frac{1}{2}}(tw)g(w)\ dw$$

$$= 2\int_{0}^{\infty} \cos(tw)\ g(w)dw,$$

that is, we can consider this integral as a Fourier cosine transformation. Let $\bar{g}(t)$ be the Fourier cosine transform of $2g(w)$ then Tranter (1956) gives the inversion formula as

$$2g(w) = \int_{0}^{\infty} \bar{g}(t)\ \cos wt\ dt.$$

If $g(w)$ is an odd an integrable function, then

$$\int_{-\infty}^{\infty} V_{-\frac{1}{2}}(tw)\; g(w)\; dw = 0;$$

it follows that for an arbitrary function g(w) the function obtained from the inversion need not be unique.

Since

$$V_{\frac{1}{2}m-1}(tw) = (\tfrac{1}{2}tw)^{1-\frac{1}{2}m}\; J_{\frac{1}{2}m-1}(tw)\; \Gamma(\tfrac{1}{2}m)$$

we have that

$$\int_{0}^{\infty} V_{\frac{1}{2}m-1}(tw)\quad \frac{2(-w^2)^{\frac{1}{2}(m-1)}}{1.3...(m-2)}\; [(\frac{d}{wdw})^{\frac{1}{2}(m-1)} g(w)]\; dw$$

$$= \int_{0}^{\infty} J_{\frac{1}{2}m-1}(tw)\,(\tfrac{1}{2}tw)^{1-\frac{1}{2}m}\Gamma(\tfrac{1}{2}m)\frac{2(-w^2)^{\frac{1}{2}(m-1)}}{1.3...(m-2)}[(\frac{d}{wdw})^{\frac{1}{2}(m-1)} g(w)]\; dw$$

$$=(\tfrac{1}{2}t)^{1-\frac{1}{2}m}\Gamma(\tfrac{1}{2}m)\int_{0}^{\infty} J_{\frac{1}{2}m-1}(tw)\;\; w\; \frac{2(-1)^{\frac{1}{2}(m-1)}}{1.3...(m-2)} w^{\frac{1}{2}m-1}[(\frac{d}{wdw})^{(\frac{1}{2}m-1)} g(w)]dw.$$

Ignoring $(\tfrac{1}{2}t)^{1-\frac{1}{2}m}$, the above integral by equation (1.5) of Tranter (1956) is the Hankel transform of

$$\frac{2(-1)^{\frac{1}{2}(m-1)}}{1.3...(m-2)}\; w^{\frac{1}{2}m-1}[(\frac{d}{wdw})^{\frac{1}{2}(m-1)} g(w)]$$

If we denote the Hankel transform of this function by $\overline{f}(t)$, then the inversion formula is given as

$$\int_0^\infty \overline{f}(t)\ t\ J_{\frac{1}{2}m-1}\ (tw)\ dt.$$

Multiplying $\overline{f}(t)$ by $\Gamma(\frac{1}{2}m)(\frac{1}{2}t)^{1-\frac{1}{2}m}$ and equating this expression to $\mathcal{E}\ V_{\frac{1}{2}m-1}(tR_n)$, we get

$$\Gamma(\tfrac{1}{2}m)\ \ (\tfrac{1}{2}t)^{1-\frac{1}{2}m}\overline{f}(t)\ =\ \left\{V_{\frac{1}{2}m-1}(t)\right\}^n=\Big[\frac{\Gamma(\frac{1}{2}m)\,J_{\frac{1}{2}m-1}(t)}{(\frac{1}{2}t)^{\frac{1}{2}m-1}}\Big]^n$$

so that

$$\overline{f}(t)\ =\ \frac{(\frac{1}{2}t)^{\frac{1}{2}m-1}}{\Gamma(\frac{1}{2}m)}\ \Big[\frac{\Gamma(\frac{1}{2}m)\ \ J_{\frac{1}{2}m-1}(t)}{(\frac{1}{2}t)^{\frac{1}{2}m-1}}\Big]^n$$

thus

$$h(w)\ =\ w^{\frac{1}{2}m}(\Gamma(\tfrac{1}{2}m))^{n-1}\int_0^\infty(\tfrac{1}{2}t)^{\frac{1}{2}m-1}t\ \ J_{\frac{1}{2}m-1}(tw)\,\Big[\frac{J_{\frac{1}{2}m-1}(t)}{(\frac{1}{2}t)^{\frac{1}{2}m-1}}\Big]^n\ dt$$

This is equation (2.2.21) of Watson (1982). Thus we have the result that

$$\frac{2(-w^2)^{\frac{1}{2}(m-1)}}{1.3...(m-2)}\Big[\big(\frac{d}{wdw}\big)^{\frac{1}{2}(m-1)}g(w)\Big]$$

$$=w^{\frac{1}{2}m}(\Gamma(\tfrac{1}{2}m))^{n-1}\int_0^\infty(\tfrac{1}{2}t)^{\frac{1}{2}m-1}t\ \ J_{\frac{1}{2}m-1}(tw)\,\Big[\frac{J_{\frac{1}{2}m-1}(t)}{(\frac{1}{2}t)^{\frac{1}{2}m-1}}\Big]^n\ dt$$

We therefore have an alternative expression for the probability distribution of R_n for odd values of m ($m\geq3$).

We now give the probability distribution function of R_n for the 2m+1 dimensional case.

Corollary: With g(w) an even function satisfying the conditions of Theorem 3, then

$$\frac{(-1)^m}{1.3.5...(2m-1)}\int_{-w}^{w} t^{2m}\left(\frac{d}{tdt}\right)^m g(t)dt$$

$$= 2G(w) - 1 + \sum_{j=0}^{m-1}(-1)^{m+j}2^{j-m+1}\frac{\Gamma(\frac{1}{2})}{\Gamma(m-j+\frac{1}{2})}\left(\frac{d}{wdw}\right)^{m-j-1}g(w),$$

where $G(w) = \int_{-w}^{w} g(w)\ dw.$

Proof: Consider the integral

$$\int_{-w}^{w} t^{2m}\left(\frac{d}{tdt}\right)^m g(t)\ dt, \qquad (3.4.8)$$

on integrating by parts we get

$$2w^{2m-1}\left(\frac{d}{wdw}\right)^{m-1}g(w)-(2m-1)\int_{-w}^{w} t^{2m-2}\left(\frac{d}{tdt}\right)^{m-1}g(t)dt \qquad (3.4.9)$$

Ignoring the constant (2m-1), then the integral in equation (3.4.9) is that of equation (3.4.8) with m replaced by (m-1). Integrating by parts again we get

$$2w^{2m-1}\left(\frac{d}{wdw}\right)^{m-1}g(w) - 2(2m-1)w^{2m-3}\left(\frac{d}{wdw}\right)^{m-2}g(w) +$$

$$2(2m-1)(2m-3)\int_{-w}^{w} t^{2m-4}\left(\frac{d}{tdt}\right)^{m-2}g(t)\ dt.$$

Repeat the operation of integrating by parts until the integral

$$(-1)^{m-1}(2m-1)(2m-3)\dots 3\int_{-w}^{w} tg^{(1)}(t)\ dt$$

is obtained. Integrating by parts we get

$$(-1)^{m-1}(2m-1)(2m-3)\dots 3\ [2wg(w)\ -\int_{-w}^{w} g(w)\ dw]$$

$$= (-1)^{m-1}(2m-1)(2m-3)\dots 3\ [2wg(w)\ +\ 1\ -\ 2\ G(w)]\,.$$

We can express $(2m-1)(2m-3)\dots(2m-2j+1)$ as

$$\frac{2^{j}\Gamma(m\ +\frac{1}{2})}{\Gamma(m\ -\ j\ +\ \frac{1}{2})}.$$

This allows us to write

$$\int_{-w}^{w} t^{2m}(\frac{d}{tdt})^{m}\ g(t)dt\ =$$

$$\sum_{j=0}^{m-1}(-1)^{j}2^{j+1}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m-j+\frac{1}{2})}(\frac{d}{wdw})^{m-j-1}g(w)+(-1)^{m-1}2^{m}\ \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})}[1-2G(w)]\,.$$

It follows that

$$\frac{(-1)^{m}}{1.3.5\dots(2m-1)}\int_{-w}^{w} t^{2m}(\frac{d}{tdt})^{m}g(t)dt\ =$$

$$2\ G(w)\ -\ 1\ +\ \sum_{j=0}^{m-1}(-1)^{m-j}2^{j-m+1}\frac{\Gamma(\frac{1}{2})}{\Gamma(m-j+\frac{1}{2})}\left(\frac{d}{wdw}\right)^{m-j-1}g(w),$$

which is the result we wanted to prove.

Denoting the distribution function of R_n by $H(r)$ and using the result that

$$H(r)\ =\ P(R_n<r)\ =\ \int_{-r}^{r}\frac{(-1)^m}{1.3...(2m-1)}\ t^{2m}\left(\frac{d}{tdt}\right)^m g(t)dt,$$

we get

$$H(r)\ =\ 2G(r)\ -\ 1\ +\ \sum_{j=0}^{m-1}(-1)^{m-j}2^{j-m+1}\frac{\Gamma(\frac{1}{2})}{\Gamma(m-j+\frac{1}{2})}\left(\frac{d}{rdr}\right)^{m-j-1}g(r).$$

Letting $\bar{H}\ =\ 1-H(r)$ and $\bar{G}(r)\ =\ 1-H(r)$, we get

$$\tfrac{1}{2}\bar{H}(r)\ =\ \bar{G}(r)\ +\ \sum_{j=0}^{m-1}(-1)^{m-j}2^{j-m}\frac{\Gamma(\frac{1}{2})}{\Gamma(m-j+\frac{1}{2})}\left(\frac{d}{rdr}\right)^{m-j-1}g(r).$$

For completeness we give a second method of determining $h(w)$ from a knowledge of $g(w)$. This method is based on a result by Watson. We do not reproduce Watson's proof here; we merely state the result and show that the two functions satisfy equation (3.4.7). In Chapter IV we state and prove a general result, which specifically includes Watson's result.

In equation (2.15), Watson (1982) gives the result

$$u_n(y) = \int_y^n g(r)h(\tfrac{y}{r})\tfrac{dr}{r},$$

where $u_n(y)$ is the sum of n independent W^{ν} random variables, $g(r)$ is the density of the resultant R_n, and $h(t)$ is the probability density of $\underline{\alpha}'\underline{U}$, where $\underline{\alpha}$ is a fixed unit vector and $\underline{U}\sim U(S_m)$.

Expressing Watson's result in our notations we write

$$g(y) = \frac{1}{B(\frac{1}{2},k)} \int_y^n (r^2-y^2)^{k-1}\frac{h(r)}{r^{2k-1}}\,dr \qquad (3.4.10)$$

where the number of dimension m is given by $m=2k+1$. Equation (3.4.6) indicates how we are to show the relationship between $h(w)$ and $g(w)$. Differentiating both sides of equation (3.4.10) with respect to y and then dividing by -y we get

$$\left(-\frac{d}{ydy}\right) g(y) = \frac{2\ \Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(k-1)} \int_y^h \frac{h(r)}{r^{2k-1}}(r^2-y^2)^{k-2}\,dr.$$

On repeating the operations of first differentiating and then dividing by -y a further (k-2) time, we get

$$\left(-\frac{d}{ydy}\right)^{k-1} g(y) = \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})}\ 2^{k-1}\int_y^n \frac{h(r)}{r^{2k-1}}\,dr.$$

Now, by using a theorem of calculus, on differentiating with respect to y we get

$$(-1)^k \frac{d}{dy}\left(\frac{d}{ydy}\right)^{k-1} g(y) = \frac{\Gamma(k+\frac{1}{2})2^{k-1}}{\Gamma(\frac{1}{2})} \frac{h(y)}{y^{2k-1}},$$

which can be written as

$$h(y) = \frac{2(-1)^k}{1.3.5...(2k-1)} y^{2k}\left(\frac{d}{ydy}\right)^k g(y);$$

this is equation (3.4.7), in which w is replaced by y.

In chapter IV we express equation (3.4.7) in a form that is suitable for computational purposes.

3.5 Some Problems

It is clear that for higher odd dimensions $(m \geq 3)$, we now have an alternative method for computing the probability density function and the probability distribution function of the length R_n of the sum of independent vectors on the unit m-sphere. How does this method compare with Watson's general result?

The Fourier inversion theorem and the convolution theorem are powerful methods in determining the probability density function of the sum of independent random variables. However, when the random variables are defined on a finite range, the probability density function of the sum consists of several components, each of which is valid for a specific subinterval of the range.

The mathematics required to evaluate the probability density function of $\sum_{i=1}^{n} W_i$ by the convolution method is rather straightforward. This exercise requires the integration of a polynomial, the identification of the subintervals, and the determination of the limits of integration.

Since the probability density function of a higher value of n is computed from that of the next lower value, great care must be taken to avoid an error in calculation, since a mistake at an earlier stage can invalidate much of the subsequent work. The degree of the polynomial which we must integrate increases as n increases and also as the number of dimensions increase. The number of subintervals increases as the values of n increase.

Watson's result of section 3·1 is quite general; here the vectors are of equal length. In Chapter IV we will present our result; which will also encompass the situation where the random vectors are of fixed but unequal length.

CHAPTER IV
SPHERICAL DISTRIBUTIONS

4.1 Introduction

Most of the literature on multivariate statistical analysis is devoted to the theory of the multivariate normal distribution. One reason that the multivariate Gaussian distribution plays such a central role in statistical analysis is that many of the results that are valid for the univariate normal distribution can be easily extended to the multivariate case as for example, the univariate central limit theorem. As in the case of the univariate normal distribution, if the statistician believes that the underlying random process is not adequately described by his multivariate Gaussian model, he must then formulate alternative distributions.

One approach is to formulate a general class of distributions that contain the multivariate normal distribution as a special case. The advantage of this approach is that many of the results which have been established for the normal distribution will hold true for members of this general class; thus, many of the statistical tests will be valid, even though the assumption of a normal distribution has not been satisfied.

One class of density functions which has received some attention is the family of spherical densities. Examples of spherical densities are

(1) the multivariate normal distribution, $\underline{X} \sim N(\underline{0}, I)$,

(2) the uniform distribution on the unit sphere $\underline{U} \sim U(S_m)$,

(3) the uniform distribution on the unit m-ball,

(4) the multivariate t distribution with n degrees of freedom and density function

$$f(\underline{x}) = \frac{\Gamma(\frac{1}{2}n+\frac{1}{2}m)}{\Gamma(\frac{1}{2}n)(n\pi)^{\frac{1}{2}m}} (1 + n\ \underline{x}'\underline{x})^{-\frac{1}{2}(n+m)},$$

and

(5) the multivariate Cauchy distribution (the multivariate t-distribution with n=1).

In section 4.2 we give some results for a random vector $\underline{X}$ having a spherical density. In sections 4.3 and 4.4, we give the relationship between the probability density function of the length of an m-component spherical vector and the probability density function of a single element of that vector. In sections 4.5, 4.6, and section 4.7, we give some examples using the multivariate normal distribution, the multivariate Cauchy distribution, the uniform distribution on the m-sphere, and the uniform distribution of the m-ball.

4.2 Some Distributional Properties

In this section we give some of the properties of a random vector having a spherical density. It is these properties which we use to develop the results in section 4.3 and 4.4.

In section 1.3, we constructed a probability space by letting

$$C_m(\underline{\zeta},t)=\left\{\underline{u}:\underline{u}\in S_m,\underline{\zeta}'\underline{x}>t\right\}$$

and letting

$$\mathcal{A}=\sigma\left\{C_m(\underline{\zeta},t):\ \underline{\zeta}\in S_m,\ t\in[-1,1]\right\}$$

be the smallest σ-algebra generated by the collection of all "caps". Define the non-negative measure U such that

$$U\left\{C_m(\underline{\zeta},\ t)\right\} = 1-A(t)=\begin{cases} 0 & 1\leq t\leq\infty \\ \displaystyle\int_t^1 \frac{(1-x^2)^{\nu-\frac{1}{2}}}{B(\frac{1}{2},\nu+\frac{1}{2})}\,dx & -1\leq t\leq 1 \\ 1 & \text{elsewhere.} \end{cases}$$

A random vector $\underline{X}$ is said to have the property of spherical symmetry if $P(\underline{\zeta}'\underline{x}\leq t)$ is independent of $\underline{\zeta}$ when $\underline{\zeta}\in S_m$. An immediate result is

Theorem 1. Let $\underline{X}$ be a random vector which is spherically symmetric, then

(i) $\frac{\underline{X}}{\|\underline{X}\|} \sim U(S_m)$ and

(ii) $\frac{\underline{X}}{\|\underline{X}\|}$ and $\|\underline{X}\|$ are stochastically independent.

Proof: Let $P(\frac{\underline{\zeta}'\underline{X}}{\|\underline{X}\|} \leq w,\ \|\underline{X}\| \leq r\} = H(w,r)$ and $I\{.\}$ be the indicator function of the event $\{.\}$. Since

$$H(w,r) = \mathcal{E}_{\underline{U}} \mathcal{E}_{\underline{X}}\ I\{\frac{\underline{u}'\underline{X}}{\|\underline{X}\|} \leq w\} . I\{\|\underline{X}\| \leq r\}$$

$$= \mathcal{E}_{\underline{X}}\ A(w) . I\{\|\underline{X}\| \leq r\}$$

$$= A(w) P(\|\underline{X}\| \leq r)$$

the result follows.

Theorem 1 gives a representation of a spherically symmetric random vector.

Corollary . If we let $\underline{X}$ be a spherically symmetric random vector then we have the following distributional result

$\underline{X} \sim R\ \underline{U}$,

where $R = \|\underline{X}\|$, $\underline{U} \sim U(S_m)$, and R and $\underline{U}$ are stochastically independent.

Some writers describe a spherically symmetric random vector by one of the following definitions:

Definition 1. If the random vectors $\underline{X}$ and $P\underline{X}$ have the same density function for all fixed orthogonal matrices P, then the vector $\underline{X}$ is said to have a spherical distribution.

Definition 2. If the probability density function of a random vector $\underline{X}$ can be written as

$$f(\underline{x}) = g(\underline{x}'\underline{x}),$$

then the vector $\underline{X}$ has a spherical distribution.

That these two definitions are equivalent should be obvious when considering Theorem 1 and Corollary 2, however we give a proof of the result.

Theorem 2. Definition 1 and Definition 2 are equivalent.

Proof: Let the probability density function of $\underline{X}$ be given by

$$f(\underline{x}) = g(\underline{x}'\underline{x}).$$

Make the transformation $\underline{Y}=P\underline{X}$, where P is a fixed orthogonal matrix such that $|P|=1$. Since the Jacobian of this transformation is unity, we obtain the density function $h(\underline{y})$ of $\underline{Y}$ as

$$h(\underline{y}) = g(\underline{y}'P'P\underline{y}) = g(\underline{y}'\underline{y});$$

thus, $\underline{X}$ and $\underline{Y}=P\underline{X}$ have the same distribution for every fixed orthogonal matrix P.

Suppose $\underline{X}$ and $P\underline{X}$ have the same probability density function for every fixed orthogonal matrix P. Denote the density function of $\underline{X}$ by $m(\underline{x})$. Make the transformation $\underline{Y}=P\underline{X}$ with P a fixed orthogonal matrix. Since $\underline{X}$ and $P\underline{X}$ have the same density function, it follows that $m(\underline{x})$ satisfies the functional equation

$$m(\underline{x}) = m(P'\underline{x}) \qquad (4.2.1)$$

for every fixed orthogonal matrix P.

If $\underline{x}=\underline{0}$ then equation (4.2.1) is true; hence, we need only examine the case of $\underline{x}\neq\underline{0}$.

Every real vector $\underline{x}$ can be written as $\|\underline{x}\|\underline{\alpha}$, where $\underline{\alpha}$ is a unit vector in the direction of $\underline{x}$. Thus the functional equation (4.2.1) is written as

$$m(\ \|\underline{x}\|\underline{\alpha}) = m(\ \|\underline{x}\|\ P'\underline{\alpha}).$$

With $\underline{x}$ fixed, P' can be a fixed orthogonal matrix, with $\underline{\alpha}$ as its first row, giving

$$m(\ \|\underline{x}\|\underline{\alpha}) = m(\ \|\underline{x}\|\underline{e}_1)$$

where $\underline{e}_1$ is a vector with unity as its first element and all other elements equal to zero. Thus, for every fixed vector we have $\underline{x}$

$$m(\|\underline{x}\|\underline{\alpha}) = m(\|\underline{x}\|\underline{e}_1)=g_*(\|\underline{x}\|),$$

giving $m(\underline{x}) = g_*(\underline{x}'\underline{x})$ for every fixed real vector $\underline{x}$. We can now conclude that the two definitions are equivalents.

These two characterization of a random vector $\underline{x}$ having a spherical distribution are useful in proving some of the distributional properties of this family of distributions.

Theorem 3. Let $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_n$ be independent, identically distributed random vectors, with $\underline{X}_1$ having a spherical distribution, then $\underline{X}_1+\underline{X}_2+\ldots+\underline{X}_n$ has a spherical distribution.

Proof: For every fixed orthogonal matrix P, we have that

$$P(\underline{X}_1+\underline{X}_2+\ldots+\underline{X}_n)= P\underline{X}_1+P\underline{X}_2+\ldots+P\underline{X}_n$$

$$= \underline{Y}_1+\underline{Y}_2+\ldots+\underline{Y}_n$$

$$\sim\underline{X}_1+\underline{X}_2+\ldots+\underline{X}_n,$$

where $\underline{Y}_i$ $i=1,2,\ldots,n$ are independent, identically distributed random vectors with $\underline{Y}_i \sim \underline{X}_i$. This completes the proof.

Theorem 4. Partition the spherical random vector $\underline{X}'$ into $(\underline{X}'_1, \underline{X}'_2)$, where $\underline{X}'_1$ is a $1 \times m_1$ vector and $\underline{X}'_2$ is a $1 \times m-m_1$ vector, then $\underline{X}'_1$ has a spherical distribution.

Proof: Our proof is complete if we can show that for every fixed orthogonal $m_1 \times m_1$ matrix Q, then $Q\underline{X}_1 \sim \underline{X}_1$. Consider the matrix P, where

$$P = \begin{bmatrix} Q & 0_1 \\ 0_1' & I \end{bmatrix},$$

where Q is any orthogonal matrix; 0_1 is an $m_1 \times m-m_1$ matrix whose elements are all zero and I is the $(m-m_1)\times(m-m_1)$ identity matrix. P is an orthogonal matrix; thus, $P\underline{X} \sim \underline{X}$ gives $Q\,\underline{X}_1 \sim \underline{X}_1$ for every fixed orthogonal matrix Q.

These two properties are useful, the first tells us that the sum of n independent random vectors, having a uniform distribution on the m-sphere, has a spherical distribution. Any result we can develop for this class of distributions is applicable to the sum of independent random vectors with a uniform distribution on the unit sphere.

The second result, Theorem 4, tells us that if $\underline{X}$ has a spherical distribution then any subvector has a spherical distribution as well. This property gives us the results in section 4.3.

It is interesting to note that we could have obtained the proof of Theorem 3 and Theorem 4 by other methods as for example, by the use of characteristic functions. We now give some further results on spherical distributions.

Theorem 5. Let $\underline{X}$ be the random vector with probability density function

$$f(\underline{x}) = c_m h(\underline{x}'\underline{x})$$

and let $\underline{U} = \frac{X}{\|\underline{X}\|}$ and $R = \|\underline{X}\|$, then

(i) R and U are independent,

(ii) $\underline{U} \sim U(S_m)$ and

(iii) the density of R is given by

$$g(r) = \frac{c_m \pi^{\frac{1}{2}m}}{\Gamma(\frac{1}{2}m)} r^{m-1} h(r) \qquad r>0.$$

Proof: In Chapter I we proved this result for the case of $\underline{X} \sim N(\underline{0}, I)$. The proof here is identical except that instead of $e^{-\frac{1}{2}\underline{x}'\underline{x}}$ we now have the function $h(\underline{x}'\underline{x})$.

Kelker (1970) partitioned $\underline{X}'$ into $(\underline{X}'_1, \underline{X}'_2)$, with $\underline{X}_1$ a qx1 component vector. He then uses the characteristic function and a polar transformation to obtain the joint density of $r_1=\|\underline{X}_1\|$ and $r_2=\|\underline{X}_2\|$ as

$$f(r_1,r_2) = \frac{4\ c_m \pi^{\frac{1}{2}m} r_1^{q-1} r_2^{m-q-1} g(r_1^2+r_2^2)}{\Gamma(\frac{1}{2}q)\Gamma(\frac{1}{2}m-\frac{1}{2}q)}.$$

The next result is taken from Lord (1954).

Theorem 6. Let $\underline{X}$ be a random vector having a spherical distribution then the characteristic function is given by

$$\phi(\rho) = \mathcal{E}\ e^{i\underline{t}'\underline{x}}$$

$$= (2\pi)^{\frac{1}{2}m} \rho^{-\frac{1}{2}m+1} \int^{\infty} r^{\frac{1}{2}m} J_{\frac{1}{2}m-1}(r\rho) g(r) dr,$$

where $\rho=\|\underline{t}\|$, $r=\|\underline{x}\|$, and $g(\|\underline{x}\|)$ is the probability density function of $\underline{X}$.

Proof: Make the transformation $\underline{U} = \frac{\underline{X}}{\|\underline{X}\|}$ and $R=\|\underline{X}\|$. The

Jacobian of this transformation is R^{m-1}. The characteristic function is given by

$$\mathcal{E}\ e^{i\underline{t}'\underline{x}} = \mathcal{E}_r \mathcal{E}_{\underline{u}}\ r^{m-1} e^{ir\underline{t}'\underline{u}}.$$

Transform $\underline{U}$ to $\underline{U}^*$, where $P\underline{U} = \underline{U}^*$and P is an orthogonal matrix with its first column being $\frac{\underline{t}}{\|\underline{t}\|}$, to get

$$\mathcal{E}_r\mathcal{E}_{\underline{u}}\ r^{m-1}e^{ir\underline{t}'\underline{u}} = \mathcal{E}_r\mathcal{E}_{u_1^*}\ r^{m-1}e^{ir\rho u_1^*},$$

where $\rho=\|\underline{t}\|$. The characteristic function of U_1^* is known and given by

$$\mathcal{E}_{u_1^*}e^{ir\rho u_1^*} = 2(\tfrac{1}{2}r\rho)^{1-\frac{1}{2}m}J_{\frac{1}{2}m-1}(r\rho)\Gamma(\tfrac{1}{2}m).$$

Using result (iii) of Theorem 5 we get

$$\phi(\rho)=\mathcal{E}_r\ 2(\tfrac{1}{2}r\rho)^{1-\frac{1}{2}m}J_{\frac{1}{2}m-1}(r\rho)\Gamma(\tfrac{1}{2}m)$$

$$= (2\pi)^{\frac{1}{2}m}\rho^{1-\frac{1}{2}m}\int_0^\infty r^{\frac{1}{2}m}J_{\frac{1}{2}m-1}(r\rho)g(r)dr,$$

which is the result we wanted to prove.

Lord points out that $\phi(\rho)$ is a function of ρ only; the same process will give the inversion formula

$$f(\underline{x})= g(r)=(2\pi)^{-\frac{1}{2}m}r^{-\frac{1}{2}m+1}\int_0^\infty \rho^{\frac{1}{2}m}J_{\frac{1}{2}m-1}(r\rho)\phi(\rho)d\rho,$$

which as Lord states is the probability density function of m variables. However we can obtain the density of R, the length of $\underline{X}$, by using a transformation.

Theorem 7. Let $\underline{X}$ be a random vector with a spherical distribution, then the probability density function of $R=\|\underline{X}\|$ is given by

$$h(r) = 2^{-\frac{1}{2}m+1}\left\{\Gamma(\tfrac{1}{2}m)\right\}^{-1} \int_0^\infty (r\rho)^{\frac{1}{2}m} J_{\frac{1}{2}m-1}(r\rho)\phi(\rho)d\rho$$

Proof: From Theorem 6 we have the result

$$\phi(t_1,t_2,\ldots,t_m) = \phi(\rho)=(2\pi)^{-\frac{1}{2}m}\rho^{-\frac{1}{2}m+1}\int_0^\infty r^{\frac{1}{2}m} J_{\frac{1}{2}m-1}(r\rho)g(r)dr.$$

Then by the Fourier inversion theorem we obtain $f(\underline{x})$ as,

$$f(\underline{x}) = \left(\frac{1}{2\pi}\right)^m \int_{R^m} e^{i\underline{t}'\underline{x}}\ \phi(\underline{t})\ dt_1\ldots dt_m.$$

Make the transformation $\underline{U} = \frac{\underline{X}}{\|\underline{X}\|}$ and $R = \|\underline{X}\|$, then

$$f(r,\ \underline{u}) = \left(\frac{1}{2\pi}\right)^m r^{m-1} \int_{R^m} e^{ir\underline{t}'\underline{u}}\phi(\underline{t})\ dt_1\ldots dt_m.$$

Transform $\underline{t}$ to $(\rho,\underline{\alpha})$, where $\underline{\alpha}=\frac{\underline{t}}{\|\underline{t}\|}$ and $\rho=\|\underline{t}\|$, then

$$f(r,\ \underline{u}) = \left(\frac{1}{2\pi}\right)^m r^{m-1}\int_0^\infty \int_{S_m} e^{i\rho r\underline{\alpha}'\underline{u}}\rho^{m-1}\ \phi(\rho)\ d\underline{\alpha}d\rho.$$

Let P be an orthogonal matrix whose first column is $\underline{u}$ then transform $\underline{\alpha}$ to $\underline{\alpha}^*=P\underline{\alpha}$; thus, the integral

$$\int_{S_m} e^{i\rho r\underline{\alpha}'\underline{u}}\, d\underline{\alpha}$$

becomes

$$\int_{S_m} e^{i\rho r\alpha_1^*}\, d\underline{\alpha}^*.$$

We can regard this integral as the characteristic function of a single element of a random vector that has a uniform distribution on the unit m-sphere, giving

$$\int_{S_m} e^{i\rho r\alpha_1^*}\, d\underline{\alpha}^* = 2\pi^{\frac{1}{2}m}\left(\tfrac{1}{2}\rho r\right)^{1-\frac{1}{2}m} J_{\frac{1}{2}m-1}(r\rho).$$

The marginal density of r is obtained by integrating the joint density f(r, $\underline{u}$)with respect to the variables of $\underline{u}$. We have

$$f(r,\ \underline{u}) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}m} \int_0^\infty (\rho r)^{1-\frac{1}{2}m} \rho^{m-1}\ J_{\frac{1}{2}m-1}(r\rho).\phi(\rho)\ d\rho.$$

Upon integrating with respect to the variable $\underline{u}$, we get

$$f(r)= \frac{r^{\frac{1}{2}m}}{2^{\frac{1}{2}m-1}\Gamma(\frac{1}{2}m)} \int_0^\infty \rho^{\frac{1}{2}m} J_{\frac{1}{2}m-1}(r\rho)\phi(\rho)d\rho.$$

This is the inversion formula for the probability density function of the random variable R, the length of a spherically symmetric random vector.

In Chapter III we mentioned Watson's result without giving a proof for it. The following is a more general statement of Watson's result and develop Watson's result as a corollary .

Theorem 8. Let T be a positive random variable such that $T \sim \left\{B(\tfrac{1}{2}k, \tfrac{1}{2}(m-k))\right\}^{\frac{1}{2}}$, and let T be independent of R, then the distribution of Z = RT is given by

$$g(z) = \frac{1}{2^{\frac{1}{2}k-1}\Gamma(\frac{1}{2}k)} \int_0^\infty r^{\frac{1}{2}k}\ z^{\frac{1}{2}k} J_{\frac{1}{2}k-1}(\rho z)\phi(\rho)d\rho,$$

where $\phi(\rho) = \mathcal{E}\ e^{i\underline{t}'\underline{x}}$ and R is the length of $\underline{X}$, a spherically symmetric random vector.

Proof: Make the transformation Z=RT and $S=T^{-2}$. The Jacobian of this transformation is seen to be $\frac{1}{2}S^{-1}$. The regions $0<R<\infty$ and $0<T<1$ are transformed into the regions $0<Z<\infty$ and $1<S<\infty$. To obtain the marginal density of Z we must integrate with respect to s.

Thus,

$$g(z) = \frac{1}{(2^{\frac{1}{2}m-1}\Gamma(\frac{1}{2}m)B(\frac{1}{2}k, \frac{1}{2}(m-k))} \int_1^\infty \int_0^\infty \rho^{\frac{1}{2}m} z^{\frac{1}{2}m}\ s^{-\frac{1}{4}m+\frac{1}{2}}(s-1)^{\frac{1}{2}(m-k-2)}$$

$$\cdot J_{\frac{1}{2}m-1}(\rho z\sqrt{s})\phi(\rho)d\rho\ ds.$$

Interchanging the order of integration and letting $\nu=\frac{1}{2}m-1$ we get

$$g(z)=\frac{1}{\left(2^{\nu}\Gamma(\nu+1)B(\frac{1}{2}k,\nu+1-\frac{1}{2}k)\right)}\int_{0}^{\infty}\phi(\rho)\rho^{\nu+1}\ z^{\nu+1}\int_{1}^{\infty}s^{-\frac{1}{2}\nu}$$

$$(s-1)^{\nu-\frac{1}{2}k}J_{\nu}(\rho z\sqrt{s})ds\ d\rho$$

But the result that

$$\int_{1}^{\infty}x^{-\frac{1}{2}\nu}(x-1)^{\mu-1}J_{\nu}(a\sqrt{x})dx = \Gamma(\mu)2^{\mu}a^{-\nu}J_{\nu-\mu}(a)$$

can be found in several texts, such as Watson's (1944). Letting $\mu = \nu-\frac{1}{2}k+1$ and $a = \rho z$ we get

$$g(z)= \frac{1}{2^{\frac{1}{2}k-1}\Gamma(\frac{1}{2}k)}\int_{0}^{\infty}\rho^{\frac{1}{2}k}z^{\frac{1}{2}k}\phi(\rho)J_{\frac{1}{2}k-1}(\rho z)d\rho .$$

the result we wanted to establish.

In Chapter I, we saw that $\underline{\alpha}'\underline{U}$ has the same distribution for every unit vector $\underline{\alpha}$ when $\underline{U}$ has a uniform distribution on the unit sphere. With this fact in mind it should be clear that the characteristic function of any subvector of $\underline{U}$ has the same form as the characteristic function of $\underline{U}$. We know that for a spherical random vector $\underline{X}$, then $\underline{X} \sim R\underline{U}$ where $R=\|\underline{X}\|$ and $\underline{U}\sim U(S_m)$. It is obvious that the characteristic function of any subvector of $\underline{X}$ has the same form as the characteristic function of $\underline{X}$. For example, if $\underline{X} \sim N(\underline{0}, I)$, then

$$\mathcal{E}\, e^{i\underline{t}'\underline{x}} = e^{-\frac{1}{2}\,\underline{t}'\underline{t}}$$

for all $\underline{t} \in S_m$.

Theorem 8 tells us that if we multiply the length of an m-component vector having a spherical distribution by a $\left\{B(\frac{1}{2}k, \frac{1}{2}(m-k)\right\}^{\frac{1}{2}}$ random variable, then the resulting random quantity has the same distribution as the length of any r-component subvector of the original vector.

Corollary . Denote the length of the m-component vector by R and its probability density function by h(r). Denote the length of any k-components of this vector by R_k and its density by $g_k(r)$ then $g_k(r)$ and h(r) satisfy the integral equation

$$g_k(r) = 2\int_r^\infty \frac{h(v)}{v^{m-2}} \frac{(v^2-r^2)^{\frac{1}{2}(m-k-2)}}{B(\frac{1}{2}k, \frac{1}{2}(m-k))} r^{k-1}\, dv \qquad (4.2.2)$$

Proof: In Theorem 8 we saw that $R_k \sim RT$ where $T \sim \left\{B(\frac{1}{2}k, \frac{1}{2}(m-k)\right\}^{\frac{1}{2}}$ and $1<k<m-1$ and is independent of R. Make the transformation Y=RT and V=R and then obtain the marginal distribution of Y by integrating the joint probability density function with respect to v.

Letting k=1 in equation (4.2.2) we get

$$g_1(r) = 2\int_r^{\infty} \frac{h(v)}{v^{m-2}} \frac{(v^2-r^2)^{\frac{1}{2}(m-3)}}{B(\frac{1}{2},\frac{1}{2}(m-1))} \, dv,$$

which is the equation found in Watson (1982).

As mentioned in our introduction, many of the results which can be proved for a random vector having a multivariate normal distribution can also be proved for a random vector having a spherical distribution. We now state and prove two such results. These results are taken from Muirhead(1982).

Theorem 9. Let $\underline{X}$ have an m-variate spherical distribution with $P(\underline{X} = \underline{0})=0$.

(i) If $W=\frac{\underline{\alpha}'\underline{X}}{\|\underline{X}\|}$ where $\underline{\alpha} \epsilon S_m$ and $\underline{\alpha}'\underline{\alpha}=1$, then

$$V = \sqrt{(m-1)}\frac{W}{\sqrt{(1-W^2)}} \quad ,$$

has a t-distribution with m-1 degrees of freedom.

(ii) If B is an mxm symmetric idempotent matrix of rank k then $S=\frac{\underline{X}'B\underline{X}}{\|\underline{X}\|^2}$ has a beta distribution with parameters $\frac{1}{2}k$ and $\frac{1}{2}(m-k)$.

Proof: Observe that in results (i) and (ii) the random variables V and S are functions of the random vector $\frac{\underline{X}}{\|\underline{X}\|}$ which we know has a uniform distribution on the unit m-sphere.

Without loss of generality we can consider $\frac{\underline{Z}}{\|\underline{Z}\|}$ where $\underline{Z}$ has a multivariate normal distribution, $\underline{Z}\sim N(\underline{0},I)$ and take $\underline{\alpha}'$ as $(1,0,\ldots,0)$. Then we can write

$$V = \frac{(m-1)^{\frac{1}{2}}\ Z_1}{\left(\sum_{i=2}^{m} Z_i^2\right)^{\frac{1}{2}}},$$

V is the ratio of a univariate N(0,1) random variable and the square root of an independent $\chi^2_{m-1}/(m-1)$ random variable; therefore, V has a t_{m-1} distribution.

To prove result (ii), note that

$$S = \frac{\underline{X}'}{\|\underline{X}\|}B\,\frac{\underline{X}}{\|\underline{X}\|},$$

and so we can again assume that

$$S = \frac{\underline{Z}'}{\|\underline{Z}\|}\,B\,\frac{\underline{Z}}{\|\underline{Z}\|}$$

where $\underline{Z}$ has a multivariate normal distribution $\underline{Z}\sim N(\underline{0},I)$. Since B is a symmetric idempotent matrix, we can find an orthogonal matrix H such that

$$HBH = \begin{bmatrix} I_k & 0 \\ 0' & 0 \end{bmatrix}$$

giving

$$S = \frac{\underline{Z}'}{\|\underline{Z}\|}B\,\frac{\underline{Z}}{\|\underline{Z}\|} = \frac{\underline{Z}'}{\|\underline{Z}\|}\,H'\;H\;B\;H'\;H\;B\;H'\;H\;\frac{\underline{Z}}{\|\underline{Z}\|}.$$

Making the transformation Y=HZ we get

$$S = \frac{\sum_{i=1}^{k} Y_i^2}{\sum_{i=1}^{m} Y_i^2}.$$

It is well known that if χ_a^2 and χ_b^2 are independent then

$$\frac{\chi_a^2}{\chi_a^2+\chi_b^2} \sim B(\tfrac{1}{2}a,\tfrac{1}{2}b)$$

which completes the proof.

This next result was given by Kariya and Eaton (1977). Here $\underline{1}=(1,1,\ldots,1)$ and $\{\underline{1}\}=\{k\underline{1}:\ k \in R^1\}$; thus, $\{\underline{1}\}$ is the span of $\underline{1}$. The proof was taken from Muirhead (1982).

Theorem 10. Let $\underline{X}$ and $\underline{Y}$ be two independent random vectors where $\underline{X}$ has an m-variate spherical distribution with $P(\underline{X}=\underline{0})=0$ and $\underline{Y}$ has any distribution whatsoever, with $P(\underline{Y}\in\underline{1})=0$. If

$$r = \frac{\sum_{i=1}^{m}(X_i-\overline{X})(Y_i-\overline{Y})}{\sqrt{\sum_{i=1}^{m}(X_i-\overline{X})^2\,\sum_{i=1}^{m}(Y_i-\overline{Y})^2}},$$

where $\overline{X} = \sum_{i=1}^{m}\frac{X_i}{m}$ and $\overline{Y} = \sum_{i=1}^{m}\frac{Y_i}{m}$ then $(m-2)^{\frac{1}{2}}\frac{r}{\sqrt{(1-r^2)}}$ has the t_{m-2} distribution, here $m>2$.

Proof: Let $M = \frac{1}{m}\,\underline{1}\;\underline{1}'$; then r can be written as

$$r = \frac{\underline{X}'(I-M)\underline{Y}}{\sqrt{[\underline{X}'(I-M)\underline{X}\;\;\underline{Y}'(I-M)\underline{Y}]}}.$$

Since I-M is idempotent of rank n-1, there is an orthogonal matrix H such that

$$H(I-M)H=\begin{bmatrix} I_{n-1} & \underline{0} \\ & \\ \underline{0}' & 0 \end{bmatrix}$$

where I_{n-1} is an identity matrix.

Put $\underline{U}=H\underline{X}$ and $\underline{V}=H\underline{Y}$ and partition $\underline{U}$ and $\underline{V}$ as

$$\underline{U} = \begin{bmatrix} \underline{U}_* \\ u_n \end{bmatrix}, \quad \underline{V} = \begin{bmatrix} \underline{V}_* \\ v_n \end{bmatrix}$$

where $\underline{U}_*$ and $\underline{V}_*$ are m-1x1 column vectors, then

$$r = \frac{\underline{U}'H(I-M)H'\underline{V}}{\sqrt{[\underline{U}'H(I-M)H'\underline{U}\;\;\underline{V}'H(I-M)H'\underline{V}]}}$$

$$= \frac{\underline{U}_*{}'\underline{V}_*}{\sqrt{[\underline{U}_*{}'\underline{U}_*V_*{}'V_*]}} \qquad = \quad \frac{\underline{U}'_*}{\|\underline{U}_*\|}\;\frac{V_*}{\|\underline{V}_*\|}.$$

Since $\underline{X}$ has a spherical distribution, we know that $\underline{U}=H\underline{X}$ has a spherical distribution and any subvector $\underline{U}_*$ of $\underline{U}$ also has a spherical distribution.

Furthermore $\underline{U}_*$ is independent of $\underline{V}_*$. Conditioning on $\underline{V}_*$ and by (i) of Theorem 11. with $\underline{\alpha}=\frac{V_*}{\|\underline{V}\|}$ then

$$(m-2)^{\frac{1}{2}}\frac{r}{\sqrt{(1-r^2)}}$$

has the t_{m-2} distribution and the proof is complete.

Since for any spherically symmetric random vector $\underline{X}$, $\underline{X}\sim R\underline{U}$ such that R and $\underline{U}$ are independently distributed with $\underline{U}\sim U(S_m)$, it is clear that any function satisfying $k(\underline{x})=k(r\underline{u})$ for $0<r<\infty$ satisfies $k(\underline{X})\sim k(\underline{U})$. Examine Theorem 9 and Theorem 10 and observe that replacing $\underline{X}$, by $R\underline{X}$ leaves the variables W and S of Theorem 9 and R of Theorem 10 unchanged.

4.3 A Distributional Property

Let Y be a random variable with probability density function g(y) satisfying

(i) $g(y) = g(-y)\geq 0$ and
(ii) $g(y) = 0 \quad y>1$,
then

$$\mathcal{E}\ e^{ity} = \mathcal{E} \sum_{k=0}^{\infty}\frac{(it)^k}{k!}\ y^j = \sum_{k=0}^{\infty}\ \frac{(it)^{2k}}{2k!}\ \mu_{2k}$$

where $\mu_{2k}=\mathcal{E}\ Y^{2k}$.

Knowing that $Y \sim W^{\nu}$, we can easily show that

$$\mu_{2k} = \frac{(2k-1)(2k-3)\ldots 3.1}{(m+2k-2)\ldots m} \qquad k = 1,2,\ldots$$

and hence obtain the characteristic function of the random variable Y as

$$\mathcal{E}\, e^{ity} = \Psi_m(it).$$

With $\underline{X}$ a random vector having a spherical distribution and with $\underline{\alpha}$ a unit vector, we have

$$\mathcal{E}\, e^{it\underline{\alpha}'\underline{x}} = \mathcal{E}\, e^{itR\underline{\alpha}'\underline{U}} = \mathcal{E}\, \Psi_m(itR)$$

$$= \int_0^{\infty} \Psi_m(itR)\, dH_m(R)$$

where $H_m(R)$ is the probability distribution function of the random variable $R=\|\underline{X}\|$. But the distribution of $\underline{\alpha}'\underline{U}$ does not depend on the unit vector $\underline{\alpha}$; therefore, $\underline{\alpha}$ may be a unit vector with each of its first k elements not equal to zero and the last (m-k) elements are all equal to zero. Partition $\underline{X}'$ into $(\underline{X}'_1, \underline{X}'_2)$ where $\underline{X}'_1$ is a 1xk row vector and $\underline{X}'_2$ is a 1xm-k row vector and partition $\underline{\alpha}'$ into $(\underline{\alpha}'_1, \underline{\alpha}'_2)$, then

$$\int_0^{\infty} \Psi_m(itR)\, dH_m(R) = \mathcal{E}\, e^{it\underline{\alpha}'\underline{x}}$$

$$= \mathcal{E}\ e^{it\underline{\alpha}'_1\underline{x}_1}$$

$$= \mathcal{E}\ e^{itR_k\underline{\alpha}'_1\underline{U}_1}$$

where $R_k = \|\underline{X}_1\|$, and $\underline{U}'_1$ is a 1xk unit vector having a uniform distribution on the k-sphere and is independent of R_k. This gives

$$\mathcal{E}\ e^{itR_k\underline{\alpha}'\underline{U}_1} = \mathcal{E}\ \Psi_k(it\ R)$$

$$= \int_0^\infty \Psi_k(it\ R)dH_k(R),$$

where $H_k(r)$ is the probability distribution function of the length of any k components of the vector $\underline{X}$.

For any m-component random vector having a spherical distribution we have the result that

$$\int_0^\infty \Psi_m(itr)dH_m(r) = \int_0^\infty \Psi_{m-1}(itr)dH_{m-1}(r) = \ldots = \int_0^\infty \Psi_1(itr)dH_1(r) \quad (4.3.1)$$

for all values of t.

Consider the function $r^{2m+1}\Psi_{2m+3}(itr)$. Differentiating this function with respect to r we observe that

$$\frac{d}{dr}\left\{r^{2m+1}\Psi_{2m+3}(irt)\right\} = (2m+1)\ r^{2m}\Psi_{2m+1}(itr);$$

thus, on integrating by parts we get

$$(2m+1)\int_0^\infty \Psi_{2m+1}(irt)h_{2m+1}(r)dr=(2m+1)\int_0^\infty r^{2m}\Psi_{2m+1}(irt)\cdot r^{-2m}h_{2m+1}(r)dr$$

$$= -\int_0^\infty r^{2m+1}\Psi_{2m+3}(irt)\frac{d}{dr}\left\{r^{-2m}h_{2m+1}(r)\right\}dr$$

for all values of t. It follows that

$$h_{2m+3}(r) = \left(\frac{1}{2m+1}\right) r^{2m+1}\frac{d}{dr}\{r^{-2m}h_{2m+1}(r)\}$$

$$= \left(\frac{2m}{2m+1}\right) h_{2m+1}(r) - \frac{r\ h^{(1)}_{2m+1}(r)}{(2m+1)} \qquad (4.3.2)$$

This equality allows us to express the density function of the length of a spherical random vector in terms of the density functions of the absolute value of a single element of that vector, if the number of dimensions is odd.

Theorem 11. Let $\underline{X}$ be a (2m+1) dimensional random vector having a spherical distribution. Denote the density function of the length of any k>1 components of the vector $\underline{X}$ by $h_k(r)$ then

$$h_{2m+1}(r) = \frac{1}{2}\sum_{j=0}^{m-1} \binom{m-1}{j}\frac{(2m-2-j)!}{(2m-1)!}(-2r)^{j+1}\ h^{(j+1)}(r) \qquad (4.3.3)$$

where $h^{(j)}(r) = \left(\frac{d}{dr}\right)^j h(r)$, and h(r) is the probability density function of a single component.

Proof: Letting m=1 in equation (4.3.3) we have

$$h_3(r) = -\, r\, h^{(1)}(r).$$

Assume the result is true for m=s. For m=s+1 we have

$$h_{2s+3}(r) = \left(\frac{2s}{2s+1}\right) h_{2s+1}(r) - \left(\frac{1}{2s+1}\right) r\, h^{(1)}_{2s+1}(r).$$

We write $r\, h^{(1)}_{2s+1}(r)$ as $rh^{(1)}_{2s+1}(r) = \frac{1}{2}\, [\frac{-2r(2s-2)!}{(2s-1)!} h^{(1)}(r) +$

$$\sum_{k=1}^{s-1} \frac{(s-1)!(2s-2-k)!}{k!(s-k)!(2s-1)!}(-2)^k(k^2+k-2s)r^{k+1}h^{(k+1)}(r) +$$

$$\binom{s-1}{s-1}\frac{(s-1)!}{(2s-1)!}(-2)^s r^{s+1}\, h^{(s+1)}(r)]$$

and $2sh_{2s+1}(r)$ as

$$\frac{1}{2}\, [2s\binom{s-1}{0}\frac{(2s-2)!}{(2s-1)!}.-2rh^{(1)}(r)+\sum_{k=1}^{s-1}2s\binom{s-1}{k}\frac{(2s-2-k)!}{(2s-1)!}(-2r)^{k+1} .h^{(k+1)}(r)].$$

On substituting for $2sh_{2s+1}(r)$ and for $rh^{(1)}_{2s+1}(r)$, we write

$$2sh_{2s+1}(r) - r\, h^{(1)}_{2s+1}(r) = \frac{1}{2}\, [\frac{(2s-1)!}{(2s-1)!} -2rh^{(1)}(r)$$

$$+ \sum_{k=1}^{s-1}[\, 2s\binom{s-1}{k}\frac{(2s-2-k)!}{(2s-1)!}(-2)^{k+1} -$$

$$\frac{(s-1)!}{k!(s-k)!(2s-1)!}(2s-2-k)!(-2)^k(k^2+k-2s)\;]r^{k+1}h^{(k+1)}(r) -$$

$$\frac{(s-1)!}{(2s-1)!}(-2)^s r^{s+1}\, h^{(s+1)}(r)] \qquad (4.3.4a)$$

The expression in the inner square brackets reduces to

$$\frac{(2s-k)!}{2s!}\binom{s}{k}(-2)^{k+1}.$$

On dividing this quantity through by (2s+1) we get

$$\frac{(2s-k)!}{(2s+1)!}\binom{s}{k}(-2)^{k+1}.$$

Using this result to simplify equation (4.3.4a) we get

$$h_{2s+3}(r) = \frac{1}{2}\sum_{j=0}^{s}\binom{s}{j} \frac{(2s-j)!}{(2s+1)!}(-2r)^{j+1}\ h^{(j+1)}(r) \qquad (4.3.4)$$

The result is true for m=s+1 and hence it is true for all positive integers m by the principle of mathematical induction.

Lemma: Let $a_k = \binom{m+1}{k}\frac{(2m+2-k)!}{(2m+2)!}\ (-2)^k$ then we have

$$(k+1)a_{k+1} + a_k = \binom{m}{k-1}\frac{(2m+1-k)!}{(2m+1)!}\ (-2)^k \quad k=m-1,\ldots,1.$$

Proof: The proof is straightforward and has been omitted.

Theorem 12. Let $H_{2m+3}(y) = \int_0^y h_{2m+3}(t)dt$, then

$$H_{2m+3}(y) = \sum_{k=0}^{m+1}\binom{m+1}{k}\frac{(2m+2-k)!}{(2m+2)!}(-2)^k y^k h^{(k-1)}(y) \qquad (4.3.5)$$

Proof: Examine equation (4.3.4), then $H_{2m+3}(y)$ is of the form

$$H_{2m+3}(y) = \frac{1}{2}\sum_{k=0}^{m+1} a_k y^k h^{(k-1)}(y)$$

where

$$h^0(y) = h(y) \text{ and } h^{(-1)}(y) = \int_0^y h(t)dt.$$

Differentiating $H_{2m+3}(y)$ we get

$$H^{(1)}_{2m+3}(y) = h_{2m+3}(y) = \frac{1}{2}\Big[\sum_{k=0}^{m+1} k a_k y^{k-1} h^{(k-1)}(y) + a_k y^k h^{(k)}(y)\Big]$$

$$= \frac{1}{2}\Big[\sum_{k=0}^{m}\Big((k+1)a_{k+1}+a_k\Big)y^k h^{(k)}y + a_{m+1}y^{m+1}h^{(m+1)}(y)\Big].$$

In Theorem 11, we established that

$$h_{2m+3}(y) = \frac{1}{2}\sum_{k=0}^{m}\binom{m}{k}\frac{(2m-k)!}{(2m+1)!}(-2y)^{k+1}h^{(k+1)}(y);$$

thus, equating the coefficient of y^{m+1}, we get

$$a_{m+1} = -\binom{m}{m}\frac{m!}{(2m+1)!}(-2)^{m+1} = \binom{m+1}{m+1}\frac{(m+1)!}{(2m+2)!}2(-2)^{m+1}.$$

Using the lemma preceding the theorem we determine the value of a_k as

$$a_k = \binom{m+1}{k}\frac{(2m+2-k)!}{(2m+2)!}2(-2)^k \qquad k=m,\ m-1,\ldots,1$$

On substituting, we see that $a_1=-2$ and thus $a_0=2$ since $a_1 + a_0 = 0$. Thus,

$$H_{2m+3}(y)= H(y) + \sum_{k=1}^{m}\binom{m+1}{k}\frac{(2m+2-k)!}{(2m+2)!}(-2y)^k h^{(k-1)}(y),$$

which is equation (4.3.5).

4.4. Solving the Integral Equation

In section 4.2 we demonstrated that the probability density function of the length of an m-component spherical vector is related to the probability density function of the absolute value of a single element of that vector by an integral equation. We are able to solve this equation both for even and odd values of m. The solution we obtained for when m is odd is identical to that given in section 4.3. We first solve this equation for odd values of m.

Theorem 13. Let $h(r)$ and $g_{2k+1}(r)$ satisfy the integral equation

$$g_{2k+1}(r) = \int_r^{\infty} \frac{h(y)(y^2-r^2)^{k-1}}{y^{2k-1}B(\frac{1}{2},k)}\,dy \tag{4.4.1}$$

where $m=2k+1$, then

$$h(r) = \frac{1}{2}\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(2k-2-j)!}{(2k-1)!}(-2r)^{j+1}\,g^{(j+1)}(y)$$

Proof: Letting $k=1$, the result is immediate on differentiating both sides of equation (4.4.1) with respect to r. Assume that the result is true for $k=s$. Now for $k=s+1$ we have

$$g_{2s+3}(r) = \int_r^\infty \frac{h(y)\,(y^2-r^2)^s}{y^{2s+1}B(\frac{1}{2},s+1)}\ dy$$

Differentiating both sides of this equation with respect to r we get

$$g^{(1)}_{2s+3}(y) = \int_r^\infty \frac{h(y)}{y^{2s+1}}\ \frac{(y^2-r^2)^{s-1}}{B(\frac{1}{2},s+1)}.-2sr\ dy,$$

with a little arithmetic this equality can be written as

$$\left(\frac{1}{2s+1}\right)\ [2sg_{2s+3}(r)-rg^{(1)}_{2s+3}(r)] = \int_r^\infty \frac{h(y)}{y^{2s-1}}\ \frac{(y^2-r^2)^{s-1}}{B(\frac{1}{2},s)}\ dy$$

Denoting $\left(\frac{1}{2s+1}\right)[2sg_{2s+3}(r)-rg^{(1)}_{2s+3}(r)\ ]$ by $g_*(r)$ we get

$$g_*(r) = \int_r^\infty \frac{h(y)\,(y^2-r^2)^{s-1}}{y^{2s-1}B(\frac{1}{2},s)}\ dy,$$

but by the induction process

$$h(r)= \tfrac{1}{2}\sum_{j=0}^{s-1}\frac{(2s-j-2)!}{(2s-1)!}\ \binom{s-1}{j}\ (-2r)^{j+1}g_*{}^{(j+1)}(r) \qquad (4.4.2)$$

The Liebenitz rule for repeated differentiation of a product allows us to write

$$g_*^{(j+1)}(r)=\frac{1}{(2s+1)}[(2s-j-1)g^{(j+1)}_{2s+3}(r) - r\, g^{(j+2)}_{2s+3}(r)];$$

on substituting for $g_*^{(j+2)}(r)$ in equation (4.4.2), we get

$$h(r)=\frac{1}{2(2s+1)}\sum_{j=0}^{s-1}\frac{(2s-j-2)!}{(2s-1)!}\binom{s-1}{j}(-2r)^{j+1}\,[(2s-j-1)g^{(j+1)}_{2s+3}(r)$$

$$-rg^{(j+2)}_{2s+3}(r)].$$

It is easy to show that the coefficient of r^{j+1} $j=0,1,...,s-1$ in the above expression is

$$\left(\frac{1}{2s+1}\right)\left(\frac{1}{2s}\right)\binom{s}{j}\frac{(2s-j)!}{(2s-1)!}(-2)^{j+1}$$

while that of r^{s+1} is

$$\frac{1}{(2s+1)!}\binom{s}{s}\, s!(-2)^{s+1},$$

giving the final equality as

$$h(r) = \tfrac{1}{2}\sum_{j=0}^{s}\frac{1}{(2s+1)!}\,(2s-j)!\,\binom{s}{j}\,(-2y)^{j+1}g^{(j+2)}_{2s+3}(r).$$

By the principle of mathematical induction, the result is true for all positive integers k.

We now investigate this integral equation for even values of m. Although, this situation is slightly more difficult than when m is odd, any elementary text on integral equations will illustrate how to solve this equation, see for example Kanwal (1971).

We need the following lemma.

Lemma. Let f(u) be a strictly increasing and differentiable function in [a,b] then

$$B(\alpha,1-\alpha) = \int_a^b \frac{f^{(1)}(u)du}{[f(u)-f(a)]^{\alpha}[f(b)-f(u)]^{1-\alpha}} \qquad 0<\alpha<1.$$

Proof: Make the transformation $y = \frac{f(u)-f(a)}{f(b)-f(a)}$. Thus,

$$\int_a^b \frac{f^{(1)}(u)du}{[fu-f(a)]^{\alpha}[f(b)-f(u)]^{1-\alpha}} = \int_0^1 y^{-\alpha}(1-y)^{\alpha-1}dy = B(1-\alpha,\alpha)$$

since $0<\alpha<1$. When m is even, let m=2k; we have the integral equation

$$g_{2k}(r) = \frac{1}{B(\frac{1}{2},\frac{1}{2}(2k-1))} \int_r^{\infty} \frac{h(r)}{r^{2k-2}} (y^2-r^2)^{\frac{1}{2}(2k-3)} dy.$$

Using the method given in section 3.4 we get the result

$$\left(\frac{-d}{rdr}\right)^{k-1} g_{2k}(r) = \frac{(2k-3)(2k-5)\ldots 3.1}{B(\frac{1}{2},\frac{1}{2}(2k-1))} \int_r^{\infty} \frac{g(y)}{y^{2k-2}} (y^2-r^2)^{-\frac{1}{2}} dy$$

that is

$$\left(\frac{-d}{rdr}\right)^{k-1} g_{2k}(r) = \frac{(k-1)!}{\pi} 2^{k-1} \int_r^{\infty} \frac{h(y)}{y^{2k-2}} (y^2-r^2)^{-\frac{1}{2}} dy.$$

Multiply both sides of this equality by $\frac{2r}{\sqrt{r^2-z^2}}$ and then integrate with respect to r to get

$$\int_z^\infty \frac{2r}{\sqrt{r^2-z^2}}\left[\left(-\frac{d}{rdr}\right)^{k+1} g_{2k}(r)\right]dr$$

$$= \frac{(k-1)!}{\pi}2^{k-1}\int_z^\infty \frac{2r}{\sqrt{r^2-z^2}} \int_r^\infty \frac{h(y)}{y^{2k-2}}(y^2-r^2)^{-\frac{1}{2}}dy\ dr.$$

Interchanging the order of integration on the right hand side of this equality we obtain

$$\int_z^\infty \frac{2r}{\sqrt{r^2-z^2}}\ \left[\left(-\frac{d}{rdr}\right)^{k+1} g_{2k}(r)\ \right]dr$$

$$= \frac{(k-1)!}{\pi}2^{k-1}\int_z^\infty \frac{h(y)}{y^{2k-2}} \int_z^y \frac{2r}{\sqrt{(r^2-z^2)(y^2-r^2)}}drdy.$$

Since $B\left(\frac{1}{2},\ \frac{1}{2}\right) = \pi$, we can write the above equality as

$$\int_z^\infty \frac{2r}{(r^2-z^2)^{\frac{1}{2}}}\left[\left(-\frac{d}{rdr}\right)^{k-1} g_{2k}(r)\right]dr = (k-1)!2^{k-1}\int_z^\infty \frac{h(y)}{y^{2k-2}}\ dy.$$

Differentiating both sides of this equality with respect to z, we get

$$h(z) = \frac{z^{2k-2}}{2^{k-1}(k-1)!}\ \frac{d}{dz}\left[\int_z^\infty \frac{2r}{(r^2-z^2)^{\frac{1}{2}}}\left[\left(-\frac{d}{rdr}\right)^{k-1} g_{2k}(r)\right]dr\right].$$

Replacing 2k by m we get

$$h(z) = \frac{z^{m-2}}{\Gamma\left(\frac{1}{2}m\right)\ 2^{\frac{1}{2}m-1}}\ \frac{d}{dz}\left[\int_z^\infty \frac{2r}{(r^2-z^2)^{\frac{1}{2}}}\left[\left(-\frac{d}{rdr}\right)^{\frac{1}{2}m-1} g_m(r)\right]dr\right]$$

Even in situations where h(z) is known, obtaining h(z) by this integral equation can be a difficult task for even values of m.

4.5 The Multivariate Normal Distribution

There are some spherical random vectors for which the probability density function of the length is known. We can check the result of section 4.3 and section 4.4 by reproducing the already known results using the methods of these two sections. We begin with the multivariate normal distribution.

Let $\underline{X}$ be a random vector such that $\underline{X} \sim N(\underline{0}, I)$ then by the result of the previous section the probability density function, say h(r), of $R = \|\underline{X}\|$ satisfies the integral equation

$$\sqrt{\frac{2}{\pi}}\; e^{-\frac{1}{2}r^2} = 2\int_r^\infty \frac{h(y)\,(y^2-r^2)^{\frac{1}{2}(m-3)}}{y^{m-2}B(\frac{1}{2},\frac{1}{2}(m-1))}\, dy \qquad (4.5.1)$$

Assume that m is odd, for convenience let m=2k+1, k=1, 2,..., then equation (4.5.1) can be written as

$$=\sqrt{\frac{2}{\pi}}\; e^{-\frac{1}{2}r^2} 2\int_r^\infty \frac{h(y)}{y^{2k-1}}\;\frac{(y^2-r^2)^{k-1}}{B(\frac{1}{2},k)}\, dy.$$

Differentiating both sides of this equality with respect to r and then dividing by -2r, we get

$$\sqrt{\frac{2}{\pi}}\; e^{-\frac{1}{2}r^2} = 2^2(k-1)!\int_r^\infty \frac{h(y)}{y^{2k-1}}\;\frac{(y^2-r^2)^{k-2}}{B(\frac{1}{2},k)}\; dy.$$

Repeat this procedure a further (k-2) time to get

$$\frac{1}{2^{k-1}}\sqrt{\tfrac{2}{\pi}}\ e^{-\frac{1}{2}r^2}=\frac{2\Gamma(k)}{B(\frac{1}{2},k)}\int_r^\infty \frac{h(y)}{y^{2k-1}}\ dy.$$

Differentiating with respect to r, we get

$$\frac{1}{2^{k-1}}\sqrt{\tfrac{2}{\pi}}\ re^{-\frac{1}{2}r^2}\ =\ \frac{2\Gamma(k)}{B(\frac{1}{2},k)}\ \frac{h(r)}{r^{2k-1}},$$

which can be written as

$$h(r)=2\frac{r^{2k}e^{-\frac{1}{2}r^2}}{2^{\frac{1}{2}(2k+1)}\Gamma\left(\frac{2k+1}{2}\right)}\qquad\qquad 0<r<\infty,$$

which is the probability density function of a χ_{2k+1} random variable.

We now examine m as an even number. Assume that m=2k, k=1,2,..., then by the discussion above

$$\sqrt{\tfrac{2}{\pi}}\ e^{-\frac{1}{2}r^2}\ =\ \frac{(2k-3)(2k-5)\ldots1}{B(\frac{1}{2},\frac{1}{2}(2k-1))}\int_r^\infty \frac{h(y)}{\sqrt{y^2-r^2}}\ \frac{1}{y^{2k-2}}\quad dy$$

$$=\pi\Gamma(k)2^{k-1}\int_r^\infty \frac{h(y)}{\sqrt{y^2-r^2}}\ \frac{1}{y^{2k-2}}\quad dy.$$

From section (4.4) we can write

$$2^k\ \Gamma(k)\ \frac{h(z)}{z^{2k-2}}=-\sqrt{\tfrac{2}{\pi}}\ \frac{d}{dz}\int_z^\infty \frac{t\ e^{-\frac{1}{2}t^2}}{\sqrt{(t^2-z^2)}}\ dt.$$

To evaluate the integral above make the transformation

$s^2=t^2-z^2$ giving $sds = tdt$ and the integral becomes

$$e^{\frac{1}{2}z^2}\int_0^\infty e^{-\frac{1}{2}s^2}ds = \sqrt{2\pi}\ e^{-\frac{1}{2}z^2};$$

we write $h(z)$ as

$$h(z) = 2\,\frac{z^{2k-1}\ e^{-\frac{1}{2}z^2}}{2^k\Gamma(k)},$$

or the density function of R as

$$h(r) = 2\,\frac{r^{2k-1}\ e^{-\frac{1}{2}r^2}}{2^k\Gamma(k)} \qquad 0<r<\infty,$$

which is the probability density of a χ_{2k} random variable.

4.6 Cauchy Distribution

We can further illustrate the use of the results of section 4.3 and section 4.4 by considering the multivariate Cauchy distribution. The density function of the length of the sum of independent Cauchy random vectors has been obtained by other methods; thus, we can compare the results of this section with other known results.

Let $\underline{Z}$ be a random vector and let the probability density function given by

$$f(\underline{z}) = \frac{\Gamma(\frac{1}{2}m+\frac{1}{2})}{(\Gamma(\frac{1}{2}))^{\frac{1}{2}(m+1)}\left(1+\underline{z}'\underline{z}\right)^{\frac{1}{2}(m+1)}},$$

then $\underline{Z}$ is said to have a multivariate Cauchy distribution. Let $\underline{Z}_1$, $\underline{Z}_2,\ldots,\underline{Z}_n$ be n independent Cauchy random vectors, and let

$$R = [(\sum_{i=1}^{n} \underline{Z}_i)'(\sum_{i=1}^{n} \underline{Z}_i)]^{\frac{1}{2}}$$

the distribution of R when n=1 is known. We now give a derivation of the probability density function of R when n≥1. First, we state and prove a result from complex analysis.

Jordan's Lemma:

Let Γ be a semicircle, centered at the origin and of radius R, and f(z) be subject to the conditions

(i) f(z) is meromorphic in the upper half plane.

(ii) $f(z) \to 0$ uniformly as $|z| \to \infty$ for $0 \leq \arg z \leq \pi$

(iii) m is positive, then

$$\int_{\Gamma} e^{miz} f(z)dz \to 0 \text{ as } R \to \infty.$$

Proof: Phillips (1956) page 123.

By (ii) of the lemma, if R is sufficiently large, we have, for all points on Γ $|f(z)|<\epsilon$. Now

$$|\exp miz| = |\exp miR\{\cos\theta + i\sin\theta\}| = \exp -mR\sin\theta.$$

Hence,

$$|\int_{\Gamma} f(z)e^{miz}\, dz| = |\int_{0}^{\pi} f(z)e^{miz}\, Re^{i\theta} i d\theta|$$

$$<\epsilon \int_{0}^{\pi} e^{-mR\ \sin\ \theta}\ Rd\theta$$

$$=2R\epsilon \int_{0}^{\pi} e^{-\ mR\ \sin\ \theta} d\theta.$$

Now it can be proved, by considering the sign of its derivative or otherwise, that $\frac{\sin\theta}{\theta}$ steadily decreases from 1 to $2/\pi$ as θ increase from 0 to $\frac{1}{2}\pi$. Hence, if $0 \leq \theta \leq \frac{1}{2}\pi$, then $\frac{\sin\theta}{\theta} \geq \frac{2}{\pi}$.

Hence

$$|\int_{\Gamma} f(z)e^{miz} dz| \ \leq\ 2R\epsilon \int_{0}^{\pi} e^{-2mR\theta} \pi d\theta$$

$$=\frac{\pi\epsilon}{m}\ (1-e^{-mR}) < \frac{\pi\epsilon}{m}.$$

from which the lemma follows.

This lemma, together with the Cauchy residue, theorem is useful in evaluating integrals of the form

$$\int_{\infty}^{\infty} Q(x)e^{imx} dx,$$

where $Q(z)=\frac{N(z)}{D(z)}$ and $N(z)$ and $D(z)$ are polynomials such that (i) $D(z)$has no real roots; (ii) the degree of $D(z)$ exceeds that of $N(z)$ by at least one and (iii) $m>0$.

Consider the following integral

$$\frac{a}{\pi}\int_{-\infty}^{\infty} \frac{e^{isx}}{a^2+x^2}\,dx.$$

When a=1 this is the characteristic function of the univariate Cauchy density function. To evaluate this integral, we take the contour which consists of the line segment [-R,R] and a semicircle, Γ, of radius R in the upper half of the (x,y) plane.

Consider the following contour integral

$$\frac{1}{\pi}\oint \frac{e^{isz}}{z^2+a^2}dz$$

where the contour is as described above. The poles of the integrand are the roots of $(z^2+a^2)=0$, that is $z=\pm ai$, but only z=ai lies within the contour. The residue at z=ai is

$$\lim_{z\to ai} (z-ai)\frac{e^{isz}}{(z-ai)(z+ai)} = \frac{e^{-sa}}{2ai}.$$

Thus the residue is $\frac{\pi e^{-s}}{a}$.

Observe that

$$\frac{a}{\pi}\oint \frac{e^{isz}}{z^2+a^2}dz = \frac{a}{\pi}\int_{-R}^{R} \frac{e^{isx}}{x^2+a^2}dx + \frac{a}{\pi}\int_{\Gamma} \frac{e^{isz}}{a^2+z^2}\,dz = e^{-sa}.$$

By Jordan's lemma the integral around Γ approaches zero. Letting R tend to infinity we get

$$\frac{a}{\pi}\int_{-\infty}^{\infty} \frac{e^{isx}}{a^2+x^2}\,dx = e^{-sa} \qquad s>0.$$

The characteristic function, $\phi(s)$, of the sum of n independent identically distributed Cauchy random variables is obtained on setting a=n; giving

$\phi(s) = e^{-ns}$.

The probability density function of the sum of n independent univariate Cauchy random variates is given by

$$f(r) = \frac{n}{\pi}\left(\frac{1}{n^2 + r^2}\right)$$

and the probability density function of the absolute value of a single element of the sum of n independent Cauchy random vectors is given by

$$h_1(r) = \frac{2n}{\pi}\left(\frac{1}{n^2+r^2}\right).$$

The techniques developed in this chapter yield the probability density function of $R=\sqrt{(\sum_{i=1}^{n}X_i)'(\sum_{i=1}^{n}X_i)]}$ for an odd number of dimensions. In the instance of 2m+1 dimension, we have the integral equation

$$\frac{2n}{\pi}\,\frac{1}{n^2+r^2} = 2\int_{r}^{\infty}\frac{h_{2m+1}(z)}{z^{2m+1}}\,\frac{(z^2-r^2)^{m-1}}{B(\frac{1}{2},m)}\,dz.$$

Differentiating both sides with respect to r we get

$$\frac{n}{\pi}\,\frac{-2r}{(n^2+r^2)^2} = \frac{-2r(m-1)}{B(\frac{1}{2},m)}\,.\int_r^\infty \frac{h_{2m+1}(z)\,(z^2-r^2)^{m-2}}{z^{2m-1}}\,dz$$

that is

$$\frac{n}{\pi}\,\frac{1}{(n^2+r^2)^2} = \frac{(m-1)}{B(\frac{1}{2},m)}\int_r^\infty \frac{h_{2m+1}(z)\,(z^2-r^2)^{m-2}}{z^{2m-1}}\,dz$$

On differentiating a further (m-2) times we get

$$\frac{n}{\pi}\,\frac{1}{(n^2+r^2)^m} = \frac{1}{B(\frac{1}{2},m)}\int_r^\infty \frac{h_{2m+1}(z)}{z^{2m-1}}\,dz.$$

Differentiating with respect to r for the final time we get

$$\frac{n}{\pi}\,.\,\frac{m.2r}{(n^2+r^2)^{m+1}} = \frac{h_{2m+1}(r)}{r^{2m-1}\,B(\frac{1}{2},m)},$$

that is

$$h_{2m+1}(r) = B(\tfrac{1}{2},m)\,.\,\frac{m\,n}{\pi}.\frac{2r^{2m}}{(n^2+r^2)^{m+1}}$$

$$=\frac{2nm}{\pi}\,\frac{\Gamma(m)\ \Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})}.\frac{r^{2m}}{(n^2+r^2)^{m+1}}$$

$$= \frac{2n\Gamma(m+1)}{\Gamma(m+\frac{1}{2})\ \Gamma(\frac{1}{2})}.\frac{r^{2m}}{(n^2+r^2)^{m+1}} \qquad 0<r<\infty.$$

We now state the theorem.

Theorem 14. Let $\underline{X}_1,\underline{X}_2,\ldots,\underline{X}_n$ be independent, identically distributed random vectors with $\underline{X}_1$ having a (2m+1) dimensional Cauchy distribution.

Let

$$R^2 = \sqrt{\left(\Sigma\underline{X}'_i\right)\left(\Sigma\underline{X}_i\right)}$$

then the density function of R is given by

$$h(r) = \frac{2n}{B(\frac{1}{2},m+\frac{1}{2})}\,\frac{r^{2m}}{(n^2+r^2)^{m+1}} \qquad 0<r<\infty.$$

For the 2m+2 dimensional case we have the integral equation

$$\frac{n}{\pi}\,\frac{1}{(n^2+r^2)} = \int_r^\infty \frac{h_{2m+2}(z)}{z^{2m}}\,\frac{(z^2-r^2)^{\frac{1}{2}(2m-1)}}{B(\frac{1}{2},\frac{1}{2}(2m+1))}\;dz.$$

Differentiating this expression m-times with respect to r we get

$$\frac{n}{(n^2+r^2)^{m+1}} = \int_r^\infty \frac{h_{2m+2}(z)}{z^{2m}}\,(z^2-r^2)^{-\frac{1}{2}}\,dz$$

giving, by the discussion in section 4.4,

$$\frac{h_{2m+2}(r)}{r^{2m}} = \frac{-n}{\pi}\,\frac{d}{dr}\int_r^\infty \frac{z}{(n^2+z^2)^{m+1}}\cdot\frac{1}{\sqrt{(z^2-r^2)}}\;dz.$$

Make the substitution that $(z^2-r^2)=t^2$, then

$$\int_r^\infty \frac{z}{(n^2+z^2)^{m+1}}\,\frac{1}{\sqrt{(z^2-r^2)}}\;dz = \int_0^\infty \frac{dt}{(n^2+t^2+r^2)^{m+1}}\;dt.$$

Make the second transformation $t=\sqrt{(n^2+r^2)}\,\tan\theta$, which gives

$$\int_0^\infty \frac{dt}{(n^2+t^2+r^2)^{m+1}} = \left(\frac{1}{n^2+r^2}\right)^{\frac{1}{2}(2m+1)} \int_0^{\pi/2} \cos^{2m}\theta \, d\theta$$

$$= \left(\frac{1}{n^2+r^2}\right)^{\frac{2m+1}{2}} B(\tfrac{1}{2},\tfrac{1}{2}(2m+1)) .$$

Thus, we have

$$\frac{h_{2m+2}(r)}{r^{2m}} = -\tfrac{n}{\pi}(2m+1) \;\; B(\tfrac{1}{2},\tfrac{1}{2}(2m+1))\frac{d}{dr}\left(\frac{1}{n^2+r^2}\right)^{-\frac{2m+1}{2}} . r$$

$$= \tfrac{n}{\pi}(2m+1)B(\tfrac{1}{2},\tfrac{1}{2}(2m+1))\left(\frac{1}{n^2+r^2}\right)^{-\frac{(2m+3)}{2}}$$

giving

$$h_{2m+2}(r) = \frac{2n}{B(\frac{1}{2},m+1)} \; \frac{r^{2m+1}}{(n^2+r^2)^{\frac{(2m+3)}{2}}} .$$

The probability density function is thus given for all value of $n \geq 1$.

We could have obtained these results by noting that an m-dimensional spherical Cauchy random vector could be obtained by considering $Z_1/Z_{m+1}, Z_2/Z_{m+1}, \ldots, Z_m/Z_{m+1}$ where Z_i $i=1,2,\ldots,m+1$ are independent $N(0,1)$ random variables. Then $(Z_1/Z_{m+1}, Z_2/Z_{m+1}, \ldots, Z_m/Z_{m+1})$ has an m-variate Cauchy distribution. Earlier, we showed by the use of characteristic function that the sum of independent Cauchy random variables has a Cauchy distribution.; If we, in fact denote this random variable by X then the probability density function of Y=X/n would be given by

$$f(y) = \frac{1}{\pi}\left(\frac{1}{1 + y^2}\right).$$

It follows that the length of the sum of independent m-variate Cauchy random vector has the same distribution as $\chi_m/n\chi_1$ where χ_1^2 and χ_m^2 are independent chi square random variables with 1 and m degrees of freedom respectively.

4.7 Uniform Distribution on the Unit Sphere

In Chapter III, we gave a detailed discussion of the sum of independent random vectors, each vector having a uniform distribution on the unit sphere. There we discussed the case of an odd number of dimensions. In Chapter IV, we gave a method which allows us to produce results whether m is even or odd.

When m is even we have that $\nu=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$ Thus, for independent, identically distributed random variables W_1, $W_2,\ldots,W_n$ with $W_1\sim W^\nu$, we must determine the probability density of $\sum_{i=1}^{n} W_i$. This is not an easy task when m is even, since the density of W_1 is no longer a polynomial of finite degree. This prevents us from obtaining the density function of $\sum_{i=1}^{n} W_i$ using the convolution theorem. Thus, the method in section 4.4 is useful only with odd values of m.

4.8 The Uniform Distribution on the M-ball

We can develop some results for a random vector having a uniform distribution on the unit m-ball by using some of the results which we proved for a random vector having a uniform distribution on the unit m-sphere. We can say that a random vector $\underline{B}$ has a uniform distribution on the unit m-ball and write $\underline{B} \sim U(B_m)$ if $P(\|\underline{B}\| \leq r) = \frac{V_m(r)}{V_m(1)} = r^m$ whenever $0 \leq r \leq 1$ and $\frac{\underline{B}}{\|\underline{B}\|} \sim U(S_m)$, where $V_m(t)$ denotes the volume of a m-dimensional ball of radius t.

Theorem 15: Let $\underline{U}$ be a random vector having a uniform distribution on the unit m-sphere and let A be a random variable such that $A \sim B(1,m)$ and independent of $\underline{U}$, then

$$\underline{B} = A\ \underline{U} \sim U(B_m).$$

Proof: The transformation $Y = a\underline{U}$ maps a point on the surface of the unit m-sphere to a point on the surface of the m-sphere of radius a. The surface area of the m-sphere of radius a is

$$\frac{2a^{m-1}(\Gamma(\frac{1}{2}))^m}{\Gamma(\frac{1}{2}m)}.$$

Thus, the conditional density of $A\underline{U}$ is given by

$$f(a\ \underline{U} \mid A = a) = \frac{\Gamma(\frac{1}{2}m)}{2a^{m-1}(\Gamma(\frac{1}{2}))^m}.$$

We obtain the density of $\underline{B} = A\underline{U}$ by multiplying conditional density given above by the marginal density of A and then integrating the expression with respect to a. Thus

$$g(\underline{B}) = \int_0^1 \frac{\Gamma(\frac{1}{2}m)\ \ m\ \ a^{m-1}}{2a^{m-1}(\Gamma(\frac{1}{2}))^m} \quad da$$

$$\int_0^1 \frac{m\Gamma(\frac{1}{2}m)}{2(\Gamma(\frac{1}{2}))^m} \quad da$$

$$= \frac{m\Gamma(\frac{1}{2}m)}{2(\Gamma(\frac{1}{2}))^m}$$

Since $0 \leq A \leq 1$, multiplication of $\underline{U}$ by A maps $\underline{U}$ onto the unit ball giving

$$f(\underline{B}) = \frac{m\Gamma(\frac{1}{2}m)}{2(\Gamma(\frac{1}{2}))^m}.$$

Theorem 16: Let $\underline{B}$ be a random vector having a uniform distribution on the unit ball, and let $\underline{\alpha}'$ be a unit vector then

$$\underline{\alpha}\ \underline{B}' \sim W^{\nu+1}.$$

Proof: There are several methods which we can use to prove the result. However, our proof is based on the results which we used in our discussion. Let

$$\phi(t) = \mathcal{E}\ e^{it\underline{\alpha}'\underline{B}}$$

$$= \mathcal{E}\ e^{itA\underline{\alpha}'\frac{\underline{Z}}{\|\underline{Z}\|}} \quad ,(\text{where } \underline{Z} \sim N(\underline{0},\ I)).$$

Thus

$$\phi(t) = \mathcal{E}\ e^{it\underline{\alpha}'\underline{B}} = \int_0^1 ma^{m-1}V_\nu(at)\ da$$

$$=\int_0^1 ma^{m-1}\Gamma(\nu+1)(\tfrac{1}{2}at)^{-\nu}\ J_\nu(at)\ da$$

$$= \left(\frac{t}{2}\right)^{-\nu}2\Gamma(\nu+2)\int_0^1 a^{\nu+1}J_\nu(at)\ da.$$

On page 373, Watson (1944) states that

$$J_{u+\nu+1}(z) = \frac{z^{\nu+1}}{2^\nu\Gamma(\nu+1)}\int_0^{\frac{1}{2}\pi} J_u(\nu\sin\theta)\mathrm{Sin}^{u+1}\theta\ \cos^{2\nu+1}\theta\ d\theta$$

which is valid when the real part of ν exceeds -1.

Make the transformation t=sinθ and then put ν=0 to get

$$\int_0^1 a^{u+1}J_u(at)da = t^{-1}J_{u+1}(t).$$

Thus we have

$$\phi(t) = \mathcal{E}\ e^{itA\underline{\alpha}'\underline{B}}$$

$$=(\tfrac{1}{2}t)^{-\nu}\ 2\Gamma(\nu+2)\ t\ J_{\nu+1}(t)$$

$$=(\tfrac{1}{2}t)^{-(\nu+1)}\Gamma(\nu+2)J_{\nu+1}(t)$$

$$=V_{\nu+1}(t)\quad = \mathcal{E}\ e^{itW}$$

where $W\sim W^{\nu+1}$.

In other words, a single element of a random vector having a uniform distribution on the unit m-ball has the same distribution as a single element of a random vector having a uniform distribution on the unit (m+2)-sphere.

We have established the relationship between $h_{2m+1}(r)$ and $h_1(r)$ for a spherical random vector, where $h_{2m+1}(r)$ is the probability density function of the resultant of the (2m+1)-component vector and $h_1(r)$ is the probability density function of the absolute value of a single element of that vector. For the (2m+1) dimensional unit sphere $h_1(r)$ is the density of

$$Y = |\sum_{i=1}^{n} W_i| ,$$

where $W_i \sim W^{\frac{1}{2}(2m+1)-1}$ and $W_1, W_2, \ldots, W_n$ are independent; whereas for the (2m+1) dimensional unit ball

$$Y = |\sum_{i=1}^{n} W_i|$$

where $W_i \sim W^{\frac{1}{2}(2m+3)-1}$.

The results of section 1.4 and section 1.2 are applicable to the case of independent random vectors, each of which has a uniform distribution on the ball.

CHAPTER V.
SUMMARY AND FUTURE RESEARCH

5.1 Summary

In Chapter I we gave some of the properties of a random vector, which is uniformly distributed on the unit m-sphere. We discussed some results which were given by Saw on the distribution of R_n, the length of the sum of n independent vectors, each of which has a uniform distribution on the unit m-sphere.

In Chapter II we dealt with the computational problems encountered in using the procedure given by Saw. A part of our investigation was to approximate the distribution of R_n using the probability density functions of a set of Beta random variables. Our results though encouraging did not improve on the method used by Saw.

In Chapter III we gave a result for the probability density of the length of the sum of independent, identically distributed random vectors, each of which have a uniform distribution on the unit m-sphere; here $m \geq 3$ is an odd integer. In Chapter IV we presented generalizations of the earlier chapters. We considered spherical random vector and derived probability density functions of their length in terms of the probability density functions of the absolute value of a single element. We also described a method for finding the density of the length for an odd number of dimensions and we were able in some situations to obtain results for even number dimensions.

5.2 Future Research.

In Chapter II a failure in a part of this study was reported. Further investigation of the problem is necessary, and it is my intention to return to this problem.

Another area of interest is that of elliptical random variables. In the same way that the family of spherical random variables contains $\underline{X} \sim N(\underline{0}, I)$ as a special case, the family of elliptical random variables contains $\underline{X} \sim N(\underline{0}, \Sigma)$ as a special case. Many of the results which have been proven for a multivariate normal distribution are also true for a spherical random vector. Although the mathematics required for an elliptical random vector are somewhat more difficult, it is my belief that we can establish some results for elliptical random vectors which are similar to results for $\underline{X} \sim N(\underline{0}, \Sigma)$. It is my intention to pursue research along these lines.

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BIOGRAPHICAL SKETCH

Everton De Courcey Rowe was born on February 21, 1956, in Checker Hall, St. Lucy, Barbados. He received his primary and secondary education in the schools of his homeland. His tertiary education was mainly at the University of the West Indies in Barbados and Trinidad.

He began his doctoral studies at the University of Florida in 1983, with an interruption of one year during this period. During his stay at the University of Florida Mr. Rowe worked as a graduate student in the Institute of Food and Agricultural Sciences and at the Biostatistic Unit. In his final year at the University of Florida he was a tutor for courses taught by Dr. Wackerly and Dr. Rosalsky.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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